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Théories de jauge et connexions généralisées sur les algébroïdes de Lie transitifs.

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au Centre de Physique Théorique - UMR 7332, de Marseille.

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to the *Centre de Physique Théorique* - UMR 7332, of Marseille.

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Résumé de la thèse

Ce mémoire de thèse rapporte les résultats obtenus au cours de mes trois années de doctorat au Centre de Physique Théorique de l'Université d'Aix-Marseille. L'objectif final de ce mémoire est la construction de théories de jauge basées sur les connexions généralisées définies sur les algébroïdes de Lie transitifs. Ce résumé suit le déroulement de la présentation de ma thèse.

Avant-propos : la physique-mathématique

La physique-mathématique est la discipline de l'étude de systèmes physiques indissociables, par nature, de leurs cadres mathématiques descriptifs. Pour la plupart des domaines de la physique, tels que la mécanique du solide, la mécanique des fluides et l'optique physique, les mathématiques sont un outil de modélisations ou d'aide à la résolution d'équations. Ainsi, la mécanique du solide est l'étude des déformations subies par un matériaux sous l'action de pressions et de forces, la mécanique des fluides étudie les phénomènes macroscopiques émergeant des interactions entre particules fluides et l'optique physique étudie des propriétés de la lumière en fonction de son milieu. Dans ces exemples, les mathématiques jouent un rôle significatif dans l'investigation scientifique, bien que l'objet de ces investigations soient extérieurs et indépendants des mathématiques qui la modélisent. A l'inverse, la relativité générale d'Einstein et la mécanique quantique sont deux exemples qui s'accordent avec la discipline de la physique-mathématique. En effet, la relativité générale d'Einstein ne peut pas être étudiée sans faire référence à la géométrie riemannienne et il n'y a pas de sens à essayer de comprendre la mécanique quantique en ignorant le formalisme des vecteurs d'états dans les espaces de Hilbert. Ces deux exemples illustrent, d'une certaine façon, une indissociabilité entre la physique et les mathématiques.

Parmi les disciplines de la physique-mathématique, nous nous intéressons aux théories de jauge associées aux groupes de symétries.

Établi dans les années 50, le modèle standard de la physique des particules repose sur deux piliers fondateurs qui sont, d'une part, l'existence de groupes de symétries agissant sur des espaces internes, et d'autre part, la construction de théories dites invariantes de jauge.

Les groupes de symétries sont associés aux forces fondamentales de la Nature. En physique des particules, les groupes de symétrie $U(1)$, $SU(2)$ et $SU(3)$ sont respectivement associés aux interactions électromagnétiques, faibles et nucléaires fortes. Ces symétries sont dites "internes" au sens où les groupes qui leurs sont associés agissent en tout point de l'espace-temps, sur des espaces abstraits ne possédant pas "d'extensions spatiales". Le cadre mathématique associé à cette formulation est le cadre de la géométrie différentielle et de la théorie des fibrés.

Par ailleurs, les théories de jauge s'intéressent aux quantités invariantes sous l'action de groupes de transformations, appelés les groupes de jauge. Une quantité présente dans une théorie de jauge est une observable physique, du moins potentiellement, si celle-ci est invariante sous l'action du groupe de jauge de la théorie.

Le point de départ de la construction décrite dans ce mémoire de thèse est de conserver le principe de jauge en tant qu'ingrédient fondamental pour la construction de théorie physique et de substituer le formalisme de la théorie des fibrés par un nouveau cadre mathématique : les algébroïdes de Lie transitifs. Le résultat de cette construction est la définition d'une théorie de jauge, qui associe à la fois la géométrie différentielle usuelle et une description algébrique nouvelle. Cette théorie donne un modèle de théorie des champs de type Yang-Mills-Higgs, décrivant la propagation de bosons vecteurs de jauge couplés à un champ tensoriel scalaire dynamique plongé, dans un potentiel quartique.

Rappels d'éléments de la géométrie différentielle

La géométrie différentielle est utilisée pour la description de calculs différentiels sur un espace topologique. Cet espace topologique ou variété est composé d'un atlas de cartes $(\mathcal{U}_i, \varphi_i)_{i\in I}$ tels que $\underset{i\in I}{\cup}\mathcal{U}_i = \mathcal{M}$ et $\varphi_i : \mathcal{U}_i \to \mathcal{O}$ soit un homéomorphisme d'un ouvert $\mathcal{U}_i$ vers un ouvert de $\mathcal{O}$ de $\mathbb{R}^m$. Un atlas permet de décrire une variété topologique de dimension m comme une collection d'espaces "plats", euclidiens ou minkowskiens, qui se recollent les uns aux autres pour former une variété globale de courbure non-nulle.

Les champs de vecteurs sur la variété assignent à chaque point p de $\mathcal{M}$ un élément X_p du plan tangent $T_p\mathcal{M}$. L'ensemble des plans tangents forme le fibré tangent $T\mathcal{M}$ et les champs de vecteurs sont alors vus comme des sections sur ce fibré. Les champs de vecteurs forment un espace vectoriel muni d'un crochet de Lie définie par le commutateur de deux champs de vecteurs. En tant qu'espace vectoriel, l'espace des champs de vecteurs se décompose, dans une carte locale $(x^1, \dots, x^m)$, sur la base $(\partial_1, \partial_2, \dots, \partial_m)$.

On définit le fibré dual $T^*\mathcal{M}$ dont les sections sont des covecteurs sur $\mathcal{M}$. Les covecteurs $\alpha \in \Gamma(T^*\mathcal{M})$ sont duaux des champs de vecteurs $X \in \Gamma(T\mathcal{M})$ dans le sens où $\alpha(X) \in C^\infty(\mathcal{M})$. Dans la même carte que précédemment, les covecteurs se décomposent sur la base $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$, où les éléments $\mathrm{d}x^\mu$ sont définis par la relation $\mathrm{d}x^\mu(\partial_\nu) = \delta^\mu_\nu$ avec δ^ν_μ le symbole de Kronecker.

D'un point de vue géométrique, l'espace des q-formes différentielles sur $\mathcal{M}$ est l'espace des sections sur le fibré dual $\wedge^q T^*\mathcal{M}$. Nous privilégions une formulation plus algébrique de ces formes, qui sont alors vues comme des applications $C^\infty(\mathcal{M})$-multilinéaires antisymétriques qui prennent pour argument r champs de vecteurs sur $\mathcal{M}$ pour donner un élément de $C^\infty(\mathcal{M})$. Les formes différentielles de degré r forment l'espace $\Omega^r(\mathcal{M})$.

Le complexe différentiel $(\Omega^\bullet(\mathcal{M}), \mathrm{d})$ est une algèbre différentielle graduée dont le complexe totale s'écrit sous la forme $\Omega^\bullet(\mathcal{M}) = C^\infty(\mathcal{M}) \oplus \Omega^1(\mathcal{M}) \oplus \Omega^2(\mathcal{M}) \oplus \dots$, et dont l'opérateur différentiel gradué $\mathrm{d} : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$ augmente le degré de forme de 1 en utilisant à la fois la représentation de $\Gamma(T\mathcal{M})$ sur $C^\infty(\mathcal{M})$ et la structure d'algèbre de Lie de $\Gamma(T\mathcal{M})$. Cette différentielle est définie pour tout $\omega \in \Omega^r(\mathcal{M})$ par la relation

$$\begin{aligned}\mathrm{d}\omega(X_1, \dots, X_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} X_i \cdot \omega(X_1, \dots, \overset{i}{\vee}, \dots, X_{r+1}) \\ + \sum_{1\leq i<j\leq r+1} \omega([X_i, X_j], X_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{r+1})\end{aligned}$$

Cette opération de dérivation définit la dérivée de Koszul sur la variété $\mathcal{M}$. Par un calcul direct, on montre que $\mathrm{d}^2 : \Omega^r(\mathcal{M}) \to \Omega^{r+2}(\mathcal{M})$ vaut 0 quelque soit le degré de forme r.

Les fibrés principaux $\mathcal{P}(\mathcal{M}, G)$ sont des fibrés au dessus de $\mathcal{M}$ dont chaque fibre est homéomorphe au groupe de structure G. Celui-ci agit "verticalement", de façon transitive, sur la fibre. Dans le cas général, un fibré principal ne peut pas s'écrire sous la forme du produit cartésien $\mathcal{P} = \mathcal{M} \times G$, à moins que celui-ci soit trivial. Cependant, localement, au dessus d'un ouvert $\mathcal{U}$ de $\mathcal{M}$, le fibré est localement trivialisable de tel sorte que $\mathcal{P} \underset{\text{loc}}{\simeq} \mathcal{U} \times G$.

Les sections sur les fibrés principaux sont des fonctions locales $s_i : \mathcal{U}_i \to \mathcal{P}$, où $\mathcal{U}_i$ désigne un ouvert de $\mathcal{M}$, et qui se recollent d'un ouvert $\mathcal{U}_i$ à un ouvert $\mathcal{U}_j$ par l'intermédiaire de fonctions de recollement $g_{ij} : \mathcal{U}_{ij} \to G$, où $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$. Dans le cas des fibrés vectoriels associés $\mathcal{E} = \mathcal{P} \times_\ell \mathcal{F}$ où $\mathcal{F}$ est un espace vectoriel et ℓ est une représentation de G sur $\mathcal{F}$, les sections donnent des applications locales $s : \mathcal{U} \to \mathcal{F}$. Ici, le formalisme en terme de sections correspond à celui des champs de jauge en théorie des champs.

En tant que variété, les fibrés principaux sont munis de champs de vecteurs $\Gamma(T\mathcal{P})$, définis comme les sections sur le fibré tangent $T\mathcal{P}$. On définit le complexe différentiel $(\Omega^\bullet(\mathcal{P}), \mathrm{d}_\mathcal{P})$ comme l'espace des formes différentielles définies sur $\mathcal{P}$ et à valeurs dans $C^\infty(\mathcal{P})$, muni de l'opérateur différentiel $\mathrm{d}_\mathcal{P} : \Omega^r(\mathcal{P}) \to \Omega^{r+1}(\mathcal{P})$, définie par la dérivée de Koszul sur $\mathcal{P}$. Soit $\mathfrak{g}$ l'algèbre de Lie du groupe de structure G, on note $(\Omega^\bullet(\mathcal{P}, \mathfrak{g}), \widehat{\mathrm{d}}_\mathcal{P})$ le complexe différentiel des formes définies sur $\mathcal{P}$ à valeurs dans l'espace tensoriel$C^\infty(\mathcal{P}) \otimes \mathfrak{g}$.

L'application $T^*\pi$ projettent les champs de vecteurs (invariants à droite) de $\mathcal{P}$ vers les champs de vecteurs sur $\mathcal{M}$. Le noyau de cette application forme les champs de vecteurs dits verticaux $\Gamma(V\mathcal{P})$ sur $\mathcal{P}$. L'appellation “verticale” provient du fait que les champs de vecteurs verticaux réalisent un déplacement infinitésimal des points de $\mathcal{P}$ le long de leur fibre associée. Alors que la “verticalité” des champs de vecteurs est correctement définie, il n'en est pas de même pour les champs de vecteurs dits “horizontaux”, qui constituent l'espace complémentaire à $\Gamma(V\mathcal{P})$ dans $\Gamma(T\mathcal{P})$, autrement dit, tels que $\Gamma(T\mathcal{P}) = \Gamma(H\mathcal{P}) \oplus \Gamma(V\mathcal{P})$, où $\Gamma(H\mathcal{P})$ désignent les champs de vecteurs horizontaux. En terme de fibré tangent à $\mathcal{P}$, on aurait $T_u\mathcal{P} = H_u\mathcal{P} \oplus V_u\mathcal{P}$ pour tout $u \in \mathcal{P}$.

Pour définire sans ambiguïté les champs de vecteurs horizontaux, on utilise une *connexion* sur $\mathcal{P}$. Du point de vue des fibrés, une connexion définie la décomposition du plan tangent $T_u\mathcal{P}$, en chaque point u, en sa partie verticale et sa partie horizontale. Elle permet également de définir les relèvements horizontaux des courbes sur $\mathcal{M}$. D'un point de vue plus algébrique, une connexion sur $\mathcal{P}$ est donnée par une 1-forme différentielle $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ définie sur $\mathcal{P}$ à valeurs dans l'algèbre de Lie $\mathfrak{g}$. On la nomme la 1-forme de connexion de Ehresmann. Celle-ci joue un rôle essentiel dans la construction de théorie invariante de jauge de type Yang-Mills. Une 1-forme de connexion réalise la décomposition des champs de vecteurs sur $\mathcal{P}$ selon $\Gamma(T\mathcal{P}) \simeq \ker(\omega) \oplus \mathrm{Im}(\omega)$ où $\ker(\omega)$ et $\mathrm{Im}(\omega)$ s'identifient aux champs de vecteurs horizontaux et verticaux, respectivement. Cette décomposition permet de définir l'opérateur $h : \Gamma(T\mathcal{P}) \to \Gamma(H\mathcal{P})$ qui projette tout champ de vecteur de $\mathcal{P}$ sur sa composante horizontale.

En utilisant un système de trivialisations locales du fibré principal, la 1-forme de connexion $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ est envoyée vers un élément $A \in \Omega^1(\mathcal{U}, \mathfrak{g})$ et définie sur $\Gamma(T\mathcal{U})$ à valeurs dans $\mathfrak{g}$. Cette 1-forme de connexion locale s'écrit dans la base $(\mathrm{d}x^\mu)_{\mu=1,\ldots,m}$ comme $A = A^a_\mu \mathrm{d}x^\mu \otimes E_a$ où $A^a_\mu \in C^\infty(\mathcal{U})$ et (E_a) désigne les éléments de la base de $\mathfrak{g}$. Dans les théories Yang-Mills, cette 1-forme de connexion décrit les bosons de jauge des théories de jauge du modèle standard. Cette construction établit les liens qui existent entre le groupe de symétrie, les connexions et les bosons médiateurs de la physique des particules.

En conservant le point de vue algébrique, on définit la courbure $R = \mathrm{d}\omega + \frac{1}{2}[\omega, \omega]$ de la connexion comme une 2-forme définie sur $\mathcal{P}$ à valeurs dans $\mathfrak{g}$. Cette courbure s'écrit localement sous la forme $R = R^a_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes E_a$ où $R_{\mu\nu} = \frac{1}{2}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + A^b_\mu A^c_\nu C^a_{bc})$ avec C^c_{ab} les constantes de structures de l'algèbre de Lie $\mathfrak{g}$. Ce terme, dit cinétique, correspond au *field strength* des théories de jauge non abéliennes. Dans le cas $G = U(1)$, le tenseur $R_{\mu\nu} \in C^\infty(\mathcal{U}) \otimes \mathbb{C}$ est le tenseur de Maxwell-Faraday de l'électromagnétisme classique.

Le groupe de jauge du fibré principal agit sur l'espace des connexions de façon *active* ou *passive*. La forme active correspond à l'action du groupe des automorphismes verticaux $A_\mu \mapsto g^{-1}A_\mu g + g^{-1}\partial_\mu g$, où $g : \mathcal{U} \to G$ est un élément du groupe de jauge. La forme passive correspond aux transformations de A_μ par changement de trivialisations de $\mathcal{P}$. Le principe de jauge appliqué aux transformations de jauge passives se traduit par le fait que les observables physiques doivent être indépendantes de la trivialisation locale de $\mathcal{P}$ choisie.

En munissant les constructions précédentes d'une métrique sur la variété et d'une métrique de Killing sur l'algèbre de Lie $\mathfrak{g}$, on définit l'opérateur de Hodge avec lequel on construit une théorie de jauge de type Yang-Mills sous la forme

$$\mathcal{L}_{YM} = (R, \star R) = \frac{1}{4}\sum_a R^a_{\mu\nu} R^{\mu\nu\, a},$$

invariante sous l'action du groupe de jauge (active *et* passive) associé au fibré principal. D'après le principe de jauge, cette théorie correspond à une "observable" en physique.[1] Elles s'interprètent comme des théories des champs décrivant la propagation des bosons de jauge non-massifs.

Mathématiques	/	Physique des particules
Connexion ω	$\leftrightarrow$	Bosons de jauge A_μ
Courbure R	$\leftrightarrow$	*Field strength* $R_{\mu\nu}$

Au final, les théories de jauge de type Yang-Mills découlent de la *théorie des connexions* sur $\mathcal{P}$. Les champs scalaires et les champs spinoriels n'émergent pas de la théorie des connexions. L'introduction de tels champs dans les théories de jauge de type Yang-Mills nécessite des structures géométriques supplémentaires. Ainsi, les champs de matière scalaire sont des sections sur un fibré associé et les spineurs sont des sections sur un fibré de Dirac.

Mathématiques	/	Physique des particules
Sections sur fibré associé	$\leftrightarrow$	Champs de matière scalaire ϕ
Sections sur fibré de Dirac	$\leftrightarrow$	Spineurs Ψ

À elles-seules, les théories Yang-Mills en permettent pas de rendre compte de l'observation des masses des bosons vecteurs $W^\pm_\mu$ et Z^0, médiateurs de l'interaction faible. Pour cela, on invoque le mécanisme de Brout-Englert-Higgs-Hagen-Guralnik-Kibble (BEHHGK, prononcé "beck") de brisure spontanée de symétrie. Le mécanisme BEHHGK consiste à coupler une théorie de jauge $U(1)\times SU(2)$ de type Yang-Mills à un champ scalaire extérieur ϕ plongé dans un potentiel scalaire quartique. En analogie avec les matériaux ferromagnétiques, ce champ est spontanément polarisé dans une direction, l'état du vide de la théorie, ce qui provoque une brisure spontanée de la symétrie $U(1) \times SU(2) \to U(1)$, ainsi que l'apparition de termes de masses sur les bosons $W^\pm$ et Z^0.

[1] La partie "observable" du Lagrangien sont les équations du mouvement du système.

A proprement parler, ce champ n'est pas issu de la théorie des connexions sur fibré principal. Il s'agit d'un champ *ad hoc* de la théorie, plongé dans un potentiel défini de façon quasi-empirique. Bien qu'à l'heure actuelle la légitimité de l'existence d'un tel champ ne fasse pratiquement plus aucun doute, le fait qu'il ne trouve pas sa place dans le cadre unificateur de la théorie des connexions peut sembler déroutant aux yeux du physicien mathématicien.

La théorie des connexions généralisées sur les algébroïdes de Lie transitifs étend le formalisme purement géométrique de la géométrie différentielle et de la théorie des connexions, en incluant des éléments algébriques. Appliquée à un modèle de théories de jauge, ces éléments algébriques amènent l'existence de champs analogues au champ scalaire du mécanisme de BEHHGK.

La théorie des algébroïdes de Lie transitifs

Les algébroïdes de Lie sont la version infinitésimale des groupoïdes de Lie. Il s'agit d'un fibré vectoriel $\mathcal{A}$ définie au dessus d'une variété $\mathcal{M}$ de dimension m, appelée la variété de base, muni d'un *ancre* $\rho : \Gamma(\mathcal{A}) \to \Gamma(T\mathcal{M})$ et équipé d'un crochet de Lie tel que, pour tout $\mathfrak{X}, \mathfrak{Y} \in \Gamma(\mathcal{A})$ et $f \in C^\infty(\mathcal{M})$, on ait

$$[\mathfrak{X}, f{\cdot}\mathfrak{Y}] = f{\cdot}[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}){\cdot}f){\cdot}\mathfrak{Y} \quad \text{et} \quad \rho([\mathfrak{X}, \mathfrak{Y}]) = [\rho(\mathfrak{X}), \rho(\mathfrak{Y})].$$

En s'inspirant du formalisme de la géométrie non-commutative, on privilégie l'aspect "section" des algébroïdes de Lie afin de se rapprocher des constructions de la théorie des champs. Ainsi, nous notons A l'espace $C^\infty(\mathcal{M})$-module des sections sur $\mathcal{A}$ *i.e.* des applications $\mathcal{M} \to \mathcal{A}$.

Si l'ancre ρ est surjectif, alors l'algébroïde de Lie est dit *transitif* et on note L le noyau de ρ, aussi appelé le noyau de A. Par commodité, on voit L comme un algébroïde de Lie qui se projette, par l'application ρ, sur 0. L'espace L s'injecte dans A par le morphisme injectif $C^\infty(\mathcal{M})$-linéaire d'algèbres de Lie $\iota : \mathsf{L} \to \mathsf{A}$. Un algébroïde de Lie transitif est défini par la suite exacte courte de $C^\infty(\mathcal{M})$-modules suivante :

$$0 \longrightarrow \mathsf{L} \xrightarrow{\iota} \mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0$$

De ce point de vue, A est une généralisation des champs de vecteurs sur $\mathcal{M}$ car il contient à la fois la géométrie des champs de vecteurs de la variété ainsi qu'un espace purement algébrique L.

Un algébroïde de Lie trivial $\mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ défini sur $\mathcal{M}$ et modélisé sur $\mathfrak{g}$ est un algébroïde de Lie transitif dont l'espace A s'écrit sous la forme de la somme directe $\mathsf{A} = \Gamma(T\mathcal{M}) \oplus \Gamma(\mathcal{M} \times \mathfrak{g})$. Le crochet de Lie s'écrit alors $[X \oplus \gamma, Y \oplus \eta] = [X, Y] \oplus (X{\cdot}\eta - Y{\cdot}\gamma + [\gamma, \eta])$ pour tout $X \oplus \gamma, Y \oplus \eta \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$. La dénomination "trivial" provient du fait que la description géométrique et algébrique de l'espace total A est ici explicite. En tant que fibré, tout algébroïde de Lie transitif est localement trivialisable, ainsi chaque élément $\mathfrak{X} \in \mathsf{A}$ peut s'écrire localement sous la forme $X \oplus \gamma$ où $X = \rho(\mathfrak{X})$ et γ est une fonction définie sur $\mathcal{U}$ à valeurs dans $\mathfrak{g}$. Un atlas d'algébroïdes de Lie est définie par la donnée, au dessus de chaque ouvert de $\mathcal{M}$, d'une trivialisation locale de A.

Un exemple classique d'algébroïde de Lie transitif est l'algébroïde de Lie d'Atiyah associé à un fibré principal $\mathcal{P}(\mathcal{M}, G)$. Ici, l'espace $\Gamma_G(\mathcal{P})$ des champs de vecteurs invariant à droite se projette sur les champs de vecteurs de $\mathcal{M}$ par l'application induite $T^*\pi$ tandis que le crochet de Lie est donné par le commutateur de deux champs de vecteurs. Le noyau de l'algébroïde de Lie d'Atiyah est l'espace $\Gamma_G(\mathcal{P}, \mathfrak{g})$ des applications G-équivariantes $v : \mathcal{P} \to \mathfrak{g}$ *i.e.* telles que $v(u \cdot g) = g^{-1}v(u)g$ pour tout $u \in \mathcal{P}, g \in G$. Géométriquement, cet espace est la version infinitésimale du groupe de jauge associé à $\mathcal{P}$. L'application injective $\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(\mathcal{P})$ est similaire à l'application qui fait correspondre à tout élément $\lambda \in \mathfrak{g}$ le vecteur verticale associé $\lambda^{\#}$.

Les complexes différentielles sur A

Les algébroïdes de Lie transitifs se représentent sur les sections d'un fibré vectoriel $\mathcal{E}$, défini au-dessus de $\mathcal{M}$ par l'application $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$, où $\mathfrak{D}(\mathcal{E})$ représente les opérateurs différentiels du premier ordre sur $\Gamma(\mathcal{E})$.

Plutôt que de définir le fibré dual $\mathcal{A}^*$, on définit les q-formes différentielles sur A en tant qu'applications $C^\infty(\mathcal{M})$-mutlilinéaire antisymétrique $\wedge^q \mathsf{A} \to \Gamma(\mathcal{E})$. On note $(\Omega^\bullet(\mathsf{A}, \mathcal{E}), \widehat{\mathrm{d}}_\phi)$ le complexe différentiel total où l'opérateur différentiel d_ϕ utilise, d'une part, la représentation ϕ de A sur $\Gamma(\mathcal{E})$ et, d'autre part, le crochet de Lie sur A. En prenant $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$ ou L, on obtient, respectivement, les complexes différentiels $(\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}})$ et $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$.

Dans l'exemple des q-formes différentielles définies sur A à valeurs dans L, on définit l'opérateur différentiel gradué $\widehat{\mathrm{d}} : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^{q+1}(\mathsf{A}, \mathsf{L})$ par la relation suivante :

$$\begin{aligned} \widehat{\mathrm{d}}\omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{r+1}) = \sum_{i=1}^{r+1} (-1)^{i+1} [\mathfrak{X}_i, \omega(\mathfrak{X}_1, \dots, \overset{i}{\vee}, \dots, \mathfrak{X}_{r+1})] \\ + \sum_{1 \leq i < j \leq r+1} \omega([\mathfrak{X}_i, \mathfrak{X}_j], \mathfrak{X}_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, \mathfrak{X}_{r+1}) \end{aligned}$$

Dans le cas des algébroïdes de Lie triviaux, les complexes différentiels $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}), \delta)$ (resp. $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$) définissent l'espace des formes différentielles définies sur $\wedge^\bullet(\Gamma(T\mathcal{M}) \oplus \Gamma(\mathcal{M} \times \mathfrak{g}))$ à valeurs dans $C^\infty(\mathcal{M})$ (resp. $\Gamma(\mathcal{M} \times \mathfrak{g})$). Les opérateurs différentiels gradués δ et $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ se décomposent en la somme de deux opérateurs différentiels

$$\delta = \mathrm{d} + s \qquad \widehat{\mathrm{d}}_{\mathsf{TLA}} = \mathrm{d} + s'$$

où d est la différentielle de de Rham, s est la différentielle de Chevalley-Eilenberg et s' est la différentielle de Chevalley-Eilenberg munie de la représentation adjointe de l'algèbre de Lie $\mathfrak{g}$ sur elle-même. Un système de trivialisations locales de A permet d'établir, localement, un isomorphisme de complexes différentiels entre $(\Omega^\bullet(\mathsf{A})|_{\mathcal{U}}, \widehat{\mathrm{d}}_{\mathsf{A}})$ (resp. $(\Omega^\bullet(\mathsf{A}, \mathsf{L})|_{\mathcal{U}}, \widehat{\mathrm{d}})$) et $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}), \delta)$ (resp. $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$).

Connexions ordinaires et connexions généralisés

Une connexion *ordinaire* sur A est un *splitting* de $C^\infty(\mathcal{M})$-modules $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$ compatible avec l'ancre ρ dans le sens $\rho \circ \nabla = \mathrm{Id}_{\Gamma(T\mathcal{M})}$.

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L} \arrow[r, "\iota"] & \mathsf{A} \arrow[r, "\rho"] & \Gamma(T\mathcal{M}) \arrow[r] \arrow[l, bend left, "\nabla"] & 0 \end{tikzcd}$$

On définit la 1-forme de connexion $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$ comme $\omega(\mathfrak{X}) = \mathfrak{X} - \nabla_{\rho(\mathfrak{X})}$ pour tout $\mathfrak{X} \in \mathsf{A}$. Par construction, cette 1-forme est normée sur L, c'est-à-dire telle que $\omega \circ \iota(\ell) = -\ell$ quelque soit $\ell \in \mathsf{L}$. Ici, le signe $-$ est conventionnel. De façon équivalente, une connexion ordinaire sur A peut être définie soit par la donnée d'une 1-forme sur A à valeurs dans L normées sur L, ou bien par la donnée d'une application ∇ tel que $\rho \circ \nabla = \mathrm{Id}_{\Gamma(T\mathcal{M})}$. L'aspect "géométrique" de la connexion sur A est ainsi transcrit sous une forme plus algébrique. Sur un algébroïde de Lie d'Atiyah associé à un fibré principal, on montre que les 1-formes de connexion normées sur L sont équivalentes aux 1-formes de connexion de Ehresmann définissant l'horizontalité des champs de vecteurs.

Localement, la 1-forme de connexion s'écrit comme $\omega_{\text{loc}} = A - \theta$ où $A \in \Omega^1(\mathcal{U}, \mathfrak{g})$ est la composante géométrique de ω, et $\theta : \Gamma(\mathcal{U} \times \mathfrak{g}) \to \Gamma(\mathcal{U} \times \mathfrak{g})$, avec $\theta(\gamma) = \gamma$ pour tout $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$, est la composante algébrique de ω. La première composante représente une 1-forme de connexion locale sur $\mathcal{U}$, la 1-forme $\theta \in \mathfrak{g}^* \otimes \mathfrak{g}$ correspond à l'expression locale de la 1-forme de Maurer-Cartan. Cette décomposition indique que les degrés de libertés de ω sont seulement portés par sa "composante" géométrique.

La dérivée covariante associée à une connexion ordinaire agit sur les sections s d'un espace de représentation de A. Elle est définie comme l'application $\mathcal{D}_\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$, donnée par $\mathcal{D}_\phi s = \mathrm{d}_\phi s + \phi(\iota \circ \omega)s$. La dérivée covariante est L-horizontale par rapport à l'opération de Cartan $(\mathsf{L}, i, \mathsf{L})$ et ne "voit" donc pas les éléments "algébriques" issus de L. Localement, cette dérivée covariante donne la dérivée covariante géométrique usuelle $\mathcal{D}_{\text{loc}} s = \mathrm{d}s + \phi(A)s$, où d désigne la dérivée de de Rham et A désigne la composante géométrique locale de ω.

La courbure associée à une connexion ordinaire ∇ sur A est définie comme l'obstruction pour ∇ d'être un morphisme d'algèbres de Lie *i.e.* $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Elle s'écrit en fonction de la 1-forme de connexion ordinaire ω comme $F = \widehat{\mathrm{d}}\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(\mathsf{A}, \mathsf{L})$ où $\widehat{\mathrm{d}}$ est la différentiel associée au complexe $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ et $[\cdot, \cdot]$ est le crochet de Lie gradué sur $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. A l'instar de la dérivée covariante, la courbure F est également L-horizontale et, en utilisant une trivialisation locale, on montre qu'elle s'écrit uniquement en fonction des composantes locales A sous la forme $F_{\text{loc}} = \mathrm{d}A + \frac{1}{2}[A, A]$ où d est la différentielle de de Rham. Les termes "algébriques" θ se simplifient, et on trouve l'expression locale de la courbure géométrique associée à une 1-forme de connexion de Ehresmann.

Une 1-forme de connexion *généralisée* ϖ sur A est une 1-forme sur A à valeurs dans L, qui n'est à priori *pas* normée sur L. Ainsi, les connexions ordinaires forment un sous-espace des connexions généralisées. On associe à toute connexion généralisée ϖ un paramètre algébrique $\tau : \mathsf{L} \to \mathsf{L}$ qui mesure l'obstruction à ce que la connexion généralisée soit une connexion ordinaire. Ce paramètre se définit, pour tout $\ell \in \mathsf{L}$, comme

$$\tau(\ell) = \varpi \circ \iota(\ell) + \ell.$$

On nomme ce paramètre le *reduced kernel endomorphism* associé à la connexion généralisé ϖ. Cet élément représente la partie "algébrique" de ϖ dans le sens où, si $\tau = 0$, alors la connexion ϖ est ordinaire, et donc géométrique. En se donnant une connexion de "référence" $\mathring{\omega}$, qui est une 1-forme de connexion ordinaire sur A, on extrait la partie "géométrique" de ϖ. Celle-ci est représentée par la connexion ordinaire *induite* associée à ϖ et définie comme $\omega = \varpi + \tau(\mathring{\omega})$. On vérifie directement que cette connexion est normée sur L. Ainsi, au moyen de $\mathring{\omega}$, toute connexion généralisée ϖ, se décompose selon (ω, τ) par la relation :

$$\varpi = \omega - \tau(\mathring{\omega})$$

La dérivée covariante $\mathcal{D}_\phi : \mathsf{A} \to \mathrm{Diff}^1(\mathcal{E})$ associée à la connexion généralisée ϖ agit sur un espace de représentation $\mathcal{E}$ de A par la relation $\widehat{\mathcal{D}}_\phi s = \mathrm{d}_\phi s + \phi \circ \iota(\varpi)s$. En prenant la décomposition (ω, τ) de ϖ, on voit que la dérivée covariante généralisée étend la dérivée covariante “géométrique” ‘$\mathcal{D}_\phi$ par la relation $\widehat{\mathcal{D}}_\phi = \mathcal{D}_\phi - \phi \circ \iota(\tau(\mathring{\omega}))$. Ainsi, dans le contexte des connexions généralisées, la dérivée covariante “usuelle” est étendue par un élément purement algébrique τ.

La courbure associée à une connexion généralisée est définie en fonction de la 1-forme ϖ comme $\widehat{F} = \widehat{\mathrm{d}}\varpi + \frac{1}{2}[\varpi, \varpi]$. En considérant la décomposition de ϖ en (ω, τ), la courbure prend une forme notablement plus complexe, qu'il est commode d'écrire comme

$$\widehat{F} = \rho^* \widehat{R} - (\rho^* \mathcal{D}\tau) \circ \mathring{\omega} + \mathring{\omega}^* R_\tau,$$

où chaque terme possède sa signification géométrique et algébrique

- Le premier terme $\rho^* \widehat{R} := F - \tau \circ \mathring{F}$, où F et $\mathring{F}$ correspondent aux courbures associées respectivement à la 1-forme de connexion ordinaire induite ω et la 1-forme de connexion de référence $\mathring{\omega}$. Par un calcul direct, on voit que $\rho^* \widehat{R}$ est L-horizontal et correspond donc à un objet géométrique défini seulement sur les champs de vecteurs de $\mathcal{M}$.

- Le second terme $(\rho^* \mathfrak{D}\tau) \circ \mathring{\omega} := [\rho^* \nabla, \tau \circ \mathring{\omega}] - \tau([\rho^* \mathring{\nabla}, \mathring{\omega}])$, où ∇ et $\mathring{\nabla}$ sont les deux connexions ordinaires sur A, données par la connexion ordinaire induite ω et la connexion ordinaire de référence $\mathring{\omega}$, respectivement. Dans ce terme se mêle une composante purement géométrique, celle liée aux connexions ordinaires, et une composante mixte, celle liée à la connexion de référence $\mathring{\omega}$. Cette connexion de référence joue un rôle important dans la description local de $\widehat{F}$.

- Le dernier terme $\mathring{\omega}^* R_\tau := \frac{1}{2}\left(\tau([\mathring{\omega}, \mathring{\omega}]) - [\tau \circ \mathring{\omega}, \tau \circ \mathring{\omega}]\right)$ s'interprète comme l'obstruction à ce que le *reduced kernel endomorphisme* τ soit un endomorphisme d'algèbre de Lie sur L.

En utilisant une trivialisation locale de A, on montre que la courbure $\widehat{F}$ peut s'écrire en fonction de la composante géométrique A de la connexion ordinaire induite ω et du champ tensoriel τ. La composant géométrique $\mathring{A}$ de la connexion de référence joue le rôle d'une connexion de fond, et ne sera pas considéré comme un champ dynamique à proprement parler. En tant qu'objet définie sur $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, la 2-forme $\widehat{F}$ se décompose sur la base $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\mathrm{loc}})$ selon

$$\widehat{F}_{\mathrm{loc}} = (\rho^* \widehat{R})^a_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes E_a + (\rho^* \mathfrak{D}\tau)^b_{\mu a}\, \mathrm{d}x^\mu \wedge \mathring{\omega}^a_{\mathrm{loc}} \otimes E_b + (R_\tau)^c_{ab}\, \mathring{\omega}^a_{\mathrm{loc}} \wedge \mathring{\omega}^b_{\mathrm{loc}} \otimes E_c$$

où le symbole $\wedge$ désigne le produit tensoriel gradué de forme, et où chaque composante $(\rho^* \widehat{R})^a_{\mu\nu}$, $(\rho^* \mathfrak{D}\tau)^b_{\mu a}$ et $(R_\tau)^c_{ab}$ sont des éléments de $C^\infty(\mathcal{U})$ correspondant aux trois termes de la courbure. Écris dans cette base, ces trois termes possèdent les “bonne lois” de recollements par changement de trivialisations.

Métrique, produit de Hodge et intégrale sur A

On souhaite définir une métrique $\widehat{g}$ sur A qui tienne compte, de façon non-naïve, de la dualité entre le côté géométrique et algébrique des algébroïdes des Lie transitifs. Soient g une métrique sur la variété de base $\mathcal{M}$ et h une métrique sur l'espace L. Une métrique $\widehat{g}$ de A est dite *inner*-non dégénérée si la métrique correspondante sur L, de la forme $h = \iota^* \widehat{g}$, est non dégénérée. On montre qu'une métrique $\widehat{g}$ est *inner*-non dégénérée si, et seulement si, il existe une unique 1-forme de connexion ordinaire $\mathring{\omega}$ (normée du L), telle que la métrique $\widehat{g}$ puisse s'écrire selon

$$\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) + h(\mathring{\omega}(\mathfrak{X}), \mathring{\omega}(\mathfrak{Y}))$$

pour tout $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$. Ainsi la donnée d'une métrique *inner* non-dégénérée sur A est équivalente à la donnée du triplet $(g, h, \mathring{\nabla})$ où $\mathring{\nabla} : \Gamma(T\mathcal{M}) \to \mathsf{A}$ est la connexion métrique sur A associée à $\mathring{\omega}$. La 1-forme de connexion métrique $\mathring{\omega}$ sert de connexion de référence pour la définition de la connexion ordinaire induite.

On utilise la 1-forme de connexion métrique $\mathring{\omega}$ comme une connexion de référence pour définir la forme volume $\omega_{\text{vol}} \in \Omega^n(\mathsf{A})$. Localement, cette forme volume s'écrit comme $\omega_{\text{vol}} \simeq \sqrt{\det(h)}\, \mathring{\omega}^1 \wedge \ldots \wedge \mathring{\omega}^n$, où h désigne la métrique sur $\Gamma(\mathcal{U} \times \mathfrak{g})$. Sur les intersections d'ouverts de $\mathcal{M}$, on s'assure que les expressions locales de ω_{vol} se recollent correctement entre elles.

Localement, la 1-forme de connexion métrique s'écrit comme $\mathring{\omega}^a_{\text{loc}} = A^a_\mu \mathrm{d}x^\mu - \theta^a$. Au lieu de décomposer localement les formes différentielles définies sur A dans la base $(\mathrm{d}x^\mu, \theta^a)$, on opte pour une décomposition dans la base $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\text{loc}})$ en utilisant la relation $\theta^a = \mathring{\omega}^a_{\text{loc}} - A^a_\mu \mathrm{d}x^\mu$. Cette nouvelle base est plus adaptée aux changements de trivialisations des composantes d'objets définies globalement sur A. Ainsi, toute q-forme différentielle, telles que $q > n$, peut s'écrire sous la forme

$$\omega_{\text{loc}} = \omega^{m.i.} \otimes \omega_{\text{vol}} + \ldots$$

où les points de suspension représentent les termes de degrés de forme algébrique strictement inférieurs à n. Par recollements, on voit que $\omega^{m.i.}$ est une $(q-n)$-forme différentielle globalement définie sur $\mathcal{M}$ à valeurs dans $C^\infty(\mathcal{M})$. La notation *m.i.* correspond à *maximal inner*.

L'intégrale *inner* définie sur $\Omega^\bullet(\mathsf{A})$ "sélectionne" le $\omega^{m.i.}$ associé à chaque élément $\Omega^q(\mathsf{A})$. Elle est définie comme :

$$\int_{inner} : \Omega^q(\mathsf{A}) \to \Omega^{q-n}(\mathcal{M}) \quad ; \quad \omega \mapsto \omega^{m.i.}$$

L'action de cette intégrale donne un résultat nul sur l'espace des formes différentielles de degré strictement inférieur à n. En composant l'intégrale *inner* avec une intégrale sur $\mathcal{M}$, définie dans le formalisme de la géométrie différentielle, on associe à tout élément de $\Omega^\bullet(\mathsf{A})$ un élément scalaire. L'intégrale sur A est définie sur les formes de $\Omega^\bullet(\mathsf{A})$ comme

$$\int_{\mathsf{A}} : \Omega^q(\mathsf{A}) \to \mathbb{R} \quad ; \quad \int_{\mathsf{A}} \omega = \int_{\mathcal{M}} \circ \int_{inner} \omega$$

Toutefois, le noyau de cette construction semble "trop gros" pour obtenir une information "suffisante" de $\Omega^\bullet(\mathsf{A})$. En effet, l'intégrale sur A donne une résultat non-nul uniquement pour les formes de degré $q = m + n$.

L'espace L est dit *orientable* si les fonctions de recollement des sections locales de la fibre $\mathcal{L}$, en tant qu'endomorphismes sur $\mathfrak{g}$, sont de déterminant strictement positifs. On dit que l'algébroïde de Lie A est *orientable* si sa variété de base $\mathcal{M}$ est orientable, ainsi que son noyau L. Associé à tout algébroïde de Lie orientable, on définit un opérateur de Hodge sur $\Omega^\bullet(\mathsf{A})$, qui réalise un isomorphisme d'espaces vectoriels $\star : \Omega^p(\mathsf{A}) \to \Omega^{m+n-p}(\mathsf{A})$, où $m = \dim(\mathcal{M})$ et $n = \dim(\mathcal{L})$, et qui se généralise facilement au complexe différentiel $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. On utilise cet opérateur de Hodge pour définir un produit scalaire sur l'espace des formes différentielles de $\Omega^\bullet(\mathsf{A})$. Soit $\omega \in \Omega^q(\mathsf{A})$ et $\eta \in \Omega^r(\mathsf{A})$, alors on a :

$$\langle \omega, \eta \rangle = \int_{\mathsf{A}} \omega \wedge \star \eta$$

Les éléments de $\Omega^\bullet(\mathsf{A})$ de degré de forme distincts *i.e.* pour lesquels $q \neq r$, sont orthogonaux entre eux par rapport à ce produit scalaire. En tenant compte de la métrique la métrique h sur L, on étend ce produit scalaire à $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ par la relation :

$$\langle \omega, \eta \rangle_h = \int_{\mathsf{A}} h(\omega, \star \eta)$$

Action infinitésimale du groupe de jauge

Sur les fibrés principaux, le groupe de jauge est défini par l'action du groupe des automorphismes verticaux. Sur les algébroïdes de Lie transitifs, le "groupe" de jauge n'existe pas, dans le cas général, et il est remplacé une action infinitésimale du noyau L.

On définit l'*action géométrique* de L sur l'espace des formes différentielles définies sur A par l'action de la dérivée de Lie associée à l'opération de Cartan $(\mathrm{L}, i, \mathsf{L})$. Toutefois, cette action géométrique, appliquée à l'espace des connexions généralisées sur A, amène des transformations confuses. Notamment, on montre que la dérivée covariante $\widehat{\mathcal{D}}_\phi$, associée à la connexion généralisée ϖ, et agissant sur un espace de représentation $\mathcal{E}$, n'est pas compatible avec la représentation de L sur cet espace *i.e.* on trouve $\widehat{\mathcal{D}}^\xi s^\xi \neq (\widehat{\mathcal{D}} s)^\xi$, où $\xi \in \mathsf{L}$ est le paramètre infinitésimal de l'action de L. Également, les transformations de jauge de la courbure de ϖ ne permettent pas de retrouver les transformations dites homogènes, associées habituellement aux courbures des connexions.

Afin de retrouver, la compatibilité de la dérivée covariante avec l'action de L ainsi que les transformations de jauge homogènes pour la courbure, on définit l'*action algébrique* de L. Par définition, cette action infinitésimale algébrique de L est représentée sur l'espace des connexions généralisées ϖ par la formule $\varpi^\xi = \varpi - [\xi, \varpi] + \widehat{\mathrm{d}}\xi$. De par cette transformation, on déduit les transformations de jauge infinitésimales de la connexion ordinaire induite ω et du morphisme τ. On trouve alors $\omega^\xi = \omega - [\xi, \omega] + \widehat{\mathrm{d}}\xi$ et $\tau^\xi = \tau - [\xi, \tau]$. La courbure associée à la connexion généralisée se transforme de façon homogène sous l'action algébrique de L comme $\widehat{F}^\xi = \widehat{F} - [\xi, \widehat{F}]$. Les actions géométriques et algébriques définissent deux représentations distinctes de L. Toutefois, elles coïncident uniquement sur l'espace des connexions ordinaires.

Sur un algébroïde de Lie d'Atiyah associé à un fibré principal $\mathcal{P}(\mathcal{M}, G)$, le groupe de jauge $\mathcal{G}$ agit "algébriquement" sur la connexion généralisée selon $\varpi^g = g^{-1}\varpi g + g^{-1}\widehat{\mathrm{d}}g$ et les transformations de jauge de la connexion ordinaire induite et du *reduced kernel endomorphism* sont données par $\omega^g = g^{-1}\omega g + g^{-1}\widehat{\mathrm{d}}g$ et $\tau^g = g^{-1}\tau g$. Ces transformations sont les versions globales des transformations obtenues par l'action algébrique infinitésimale de L.

Théories de jauge de type Yang-Mills-Higgs

On construit une théorie de jauge à partir d'une connexion généralisée sur un algébroïde de Lie transitif orientable, muni d'une métrique *inner*-non dégénérée $\widehat{g} = (g, h, \mathring{\nabla})$, dont la métrique *inner* h est une métrique de Killing sur L. L'action fonctionnelle $\mathcal{S}_{gauge}[\varpi]$ est définit comme la "norme" de la courbure associée à la connexion généralisée ϖ :

$$\mathcal{S}_{Gauge}[\varpi] = \langle \widehat{F}, \widehat{F} \rangle_h$$

Sous cette forme compacte, on voit directement que cette action est invariante sous l'action algébrique infinitésimale de L.

Les théories de jauge de la physique des particules sont usuellement décrites en fonction du lagrangien associé à l'action. Ici, on définit la densité lagrangienne (ou simplement lagrangien) $\mathcal{L}[\varpi]$ par la relation $\mathcal{L}[\varpi]\mathrm{dvol} = \int_{inner} h(\widehat{F}, \star\widehat{F})$, où dvol désigne la forme volume sur $\mathcal{M}$. Pour obtenir une expression plus explicite de la théorie de jauge correspondant à ce lagrangien, on écrit la courbure $\widehat{F}$ en fonctions des champs de jauge A_μ, associés à la connexion ordinaire induite, et des champs tensoriels scalaires τ_a^b, issues de le composante algébrique de ϖ. En appliquant directement la définition du produit scalaire $\langle\cdot,\cdot\rangle_h$, on trouve :

$$\begin{aligned}\mathcal{L}[A,\tau] = {} & \lambda_1\ g^{\mu_1\mu_2} g^{\nu_1\nu_2}\ h_{a_1a_2} \\ & \Big(\partial_{\mu_1}A^{a_1}_{\nu_1} - \partial_{\nu_1}A^{a_1}_{\mu_1} + A^{b_1}_{\mu_1}A^{c_1}_{\nu_1}C^{a_1}_{b_1c_1} - \tau^{a_1}_{b_1}\Big(\partial_{\mu_1}\mathring{A}^{b_1}_{\nu_1} - \partial_{\nu_1}\mathring{A}^{b_1}_{\mu_1} + \mathring{A}^{d_1}_{\mu_1}\mathring{A}^{e_1}_{\nu_1}C^{b_1}_{d_1e_1}\Big)\Big)\cdot \\ & \Big(\partial_{\mu_2}A^{a_2}_{\nu_2} - \partial_{\nu_2}A^{a_2}_{\mu_2} + A^{b}_{\mu_2}A^{c}_{\nu_2}C^{a_2}_{bc} - \tau^{a_2}_{b_2}\Big(\partial_{\mu_2}\mathring{A}^{b_2}_{\nu_2} - \partial_{\nu_2}\mathring{A}^{b_2}_{\mu_2} + \mathring{A}^{d_1}_{\mu_2}\mathring{A}^{e_1}_{\nu_2}C^{b_2}_{d_1e_1}\Big)\Big) \\ & + \lambda_2\ g^{\mu_2\mu_1} h^{a_2a_1} h_{b_1,b_2} \\ & \Big(\partial_{\mu_1}\tau^{b_1}_{a_1} + A^{c_1}_{\mu_1}\tau^{d_1}_{a_1}C^{b_1}_{c_1d_1} - \mathring{A}^{c_1}_{\mu_1}\tau^{b_1}_{d_1}C^{d_1}_{c_1a_1}\Big)\cdot\Big(\partial_{\mu_2}\tau^{b_2}_{a_2} + A^{c_2}_{\mu_2}\tau^{d_2}_{a_2}C^{b_2}_{c_2d_2} - \mathring{A}^{c_2}_{\mu_2}\tau^{b_2}_{d_2}C^{d_2}_{c_2a_2}\Big) \\ & + \lambda_3 h_{c_1c_2}h^{a_1a_2}h^{b_1b_2}\ \Big(\tau^{c_1}_{d_1}C^{d_1}_{a_1b_1} - \tau^{d_1}_{a_1}\tau^{e_1}_{b_1}C^{c_1}_{d_1e_1}\Big)\cdot\Big(\tau^{c_2}_{d_2}C^{d_2}_{a_2b_2} - \tau^{d_2}_{a_2}\tau^{e_2}_{b_2}C^{c_2}_{d_2e_2}\Big)\end{aligned}$$

où $\lambda_1 = \frac{1}{4m(m-1)}$, $\lambda_2 = \frac{1}{mn}$ et $\lambda_3 = \frac{1}{4n(n-1)}$ sont des coefficients combinatoires et $C^c_{ab} \in \mathbb{R}$ sont les constantes de structures de l'algèbre de Lie $\mathfrak{g}$, dont on choisit la base de telle sorte que ces constantes soient réelles.

Le premier terme, factorisé par λ_1, représente la partie Yang-Mills de la théorie. C'est le terme cinétique, défini comme le "carré" du *field strength* associé aux bosons de jauge A_μ. La composante géométrique $\mathring{A}_\mu$ de la connexion métrique joue le rôle d'une connexion de fond, habituel en théorie des champs. Usuellement, ce terme est mis à zéro (possible seulement localement) et on obtient alors le terme Yang-Mills usuel. Le second terme est la dérivée covariante du champ tensoriel scalaire τ_a^b, qui s'interprète en théorie des champs comme un couplage minimal entre les bosons de jauge A_μ et le champ scalaire τ_a^b. Ici encore, la description de la dérivée covariante est donnée en présence de la connexion de fond $\mathring{A}_\mu$. Le dernier terme est un terme potentiel quartique dans lequel est plongé le champ tensoriel scalaire τ_a^b. Ce terme potentiel possède une interaction algébrique univoque : il s'agit de l'obstruction pour le paramètre τ de préserver le crochet de Lie sur L. Le résultat finale est donc une théorie de jauge de type Yang-Mills-Higgs décrivant des bosons de jauge A_μ en interaction avec une "famille" de champ scalaire τ_a^b.

En prenant le paramètre τ égal à 0, autrement dit en considérant l'espace des connexions ordinaires, normées sur L, la théorie se ramène au "carré" de la courbure de la composante géométrique de ω.

$$\begin{aligned}\mathcal{L}[A,\tau=0] = \lambda_1\ g^{\mu_1\mu_2}g^{\nu_1\nu_2}\ h_{a_1a_2}\Big(\partial_{\mu_1}A^{a_1}_{\nu_1} - \partial_{\nu_1}A^{a_1}_{\mu_1} + A^{b_1}_{\mu_1}A^{c_1}_{\nu_1}C^{a_1}_{b_1c_1}\Big)\cdot \\ \Big(\partial_{\mu_2}A^{a_2}_{\nu_2} - \partial_{\nu_2}A^{a_2}_{\mu_2} + A^{b}_{\mu_2}A^{c}_{\nu_2}C^{a_2}_{bc}\Big)\end{aligned}$$

On obtient alors une théorie de jauge non-abélienne de type Yang-Mills qui décrit la propagation de bosons de jauge non-massifs.

Dans l'esprit de la théorie des champs, on identifie les sous-espaces de $\Omega^1(\mathsf{A},\mathsf{L})$ qui correspondent à des espaces de "solutions" de la théorie, c'est-à-dire pour lesquels le terme potentiel est minimal, c'est-à-dire nul. Cette identification est similaire au “choix du vide” pour le champ scalaire ϕ du mécanisme de BEHHGK. Ici, le potentiel est minimal si, et seulement si, le paramètre τ de la théorie est un endomorphisme d'algèbres de Lie sur L. De ce point de vue, les théories Yang-Mills usuelles de la théorie des champs, ici associées à $\tau = 0$, correspondent à une classe de solutions d'une théorie des champs plus générale, basée sur les connexions généralisées sur les algébroïdes de Lie transitifs. Également, on montre que cet espace de solution est préservé par l'action algébrique (et géométrique) de L.

L'espace des connexions généralisées dont le champ tensoriel scalaire associé à la composante algébrique de ϖ est de la forme $\tau_a^b = \delta_a^b$ (ou encore $\tau = \mathrm{Id}_{\mathsf{L}}$) correspond à une seconde classe de solution de la théorie. Celle-ci fait apparaître des termes de masse pour les champs de jauge A_μ, écrits sous la forme $m_{ab}A_\mu^a A^{\mu\, b}$, où la matrice de masse est fonction des dimensions m et n, de la métrique h et des constantes de structures C_{ab}^c et qui s'écrit comme :

$$m_{ab} = \frac{1}{mn}\, h^{a_1 a_2} h_{b_1 b_2} C_{a a_1}^{b_1} C_{b a_2}^{b_2}$$

On suppose que la métrique h est localement constante *i.e.* $h_a^b \in \mathbb{R}$ pour tout $a, b = 1, \dots, n$, de telle sorte que ce terme de masse ne dépende pas du point. Ainsi, ce sous-espace de solutions, décrit la propagation de champs vectoriel massifs A_μ. Toutefois, ce sous-espace n'est pas invariant sous l'action de jauge de L.

Nouvelle méthode de brisure de symétrie.

Dans le mécanisme de BEHHGK, le champ scalaire ϕ est introduit à la main dans la théorie. Il est plongé dans un potentiel (dont la forme varie en fonction de l'énergie) et se polarise spontanément dans un état de vide qui minimise ce terme potentiel. Le groupe de symétrie initial de la théorie est alors spontanément réduit à un de ses sous-groupes, celui-ci laissant invariant cet état de vide. Les perturbations du champ ϕ autour de l'état de vide donnent les bosons de Higgs de la théorie.

Dans notre modèle, le champ scalaire ϕ est remplacé par un champ tensoriel scalaire τ_a^b, spécifique à l'espace des connexions généralisées sur A. Le potentiel quartique associé à ce champ découle également du formalisme des algébroïdes de Lie transitifs. Il est défini sous une forme algébrique et sa “forme” ne varie pas en fonction des paramètres libres de la théorie. Cet écart par rapport au modèle usuel de brisure de symétrie incite à substituer la méthode de “polarisation spontanée” du champ τ_a^b par une nouvelle méthode de réduction de symétrie.

Dans l'esprit du mécanisme de Goldstone de la théorie des champs, il est possible de “transfèrer” une partie des degrés de liberté de τ vers les bosons de jauge A_μ, ce afin de construire des champs composites invariant sous l'action du groupe de jauge. Cette méthode est illustrée dans le cas des algébroïdes de Lie d'Atiyah.

Sur un algébroïde de Lie d'Atiyah associé à un fibré principal $\mathcal{P}(\mathcal{M}, G)$, on considère l'espace $\mathcal{A}_{\mathrm{Id}_{\mathsf{L}}}$ des connexions généralisées sur $\Gamma_G(\mathcal{P})$ dont le *reduced kernel endomorphism* τ est de la forme $\tau(\ell) = \mathrm{Ad}_{u^{-1}}\ell$, pour tout $\ell \in \mathsf{L}$, où $u : \mathcal{P} \to G$ est un élément du groupe de jauge $\mathcal{G}$. En prenant G un groupe de Lie simple, alors les degrés de libertés dynamiques de jauge de τ sont tous contenus dans le champ u, ceux-ci se transformant sous l'action de $\mathcal{G}$ de jauge comme $u^g = g^{-1}u$.

Dans l'espace fonctionnel des champs de la théorie, on réalise le changement de variables $(A, \tau = \mathrm{Ad}_{u^{-1}}) \leftrightarrow (\widehat{A}, u)$ où $\widehat{A} = u^{-1}Au + u^{-1}\mathrm{d}u$ est un champ composite, invariant sous l'action du groupe de jauge. En utilisant la métrique de Killing h de la théorie, on montre que le lagrangien $\mathcal{L}[A,\tau]$ s'écrit uniquement en fonction des champs $\widehat{A}$, le champ u n'apparaissant plus explicitement dans la théorie dès lors, celui-ci n'est plus un champ dynamique de la théorie. Dans les nouvelles variables, le lagrangien $\mathcal{L}[\widehat{A}]$ s'écrit donc

$$\begin{aligned}\mathcal{L}[\widehat{A},\widehat{s}] = \lambda_1\ & g^{\mu_1\mu_2}g^{\nu_1\nu_2}\ h_{a_1a_2} \\ & \Big(\partial_{\mu_1}\widehat{A}^{a_1}_{\nu_1} - \partial_{\nu_1}\widehat{A}^{a_1}_{\mu_1} + \widehat{A}^{b_1}_{\mu_1}\widehat{A}^{c_1}_{\nu_1}C^{a_1}_{b_1c_1} - \partial_{\mu_1}\mathring{A}^{a_1}_{\nu_1} + \partial_{\nu_1}\mathring{A}^{a_1}_{\mu_1} - \mathring{A}^{d_1}_{\mu_1}\mathring{A}^{e_1}_{\nu_1}C^{a_1}_{d_1e_1}\Big)\cdot \\ & \Big(\partial_{\mu_2}\widehat{A}^{a_2}_{\nu_2} - \partial_{\nu_2}\widehat{A}^{a_2}_{\mu_2} + \widehat{A}^{b}_{\mu_2}\widehat{A}^{c}_{\nu_2}C^{a_2}_{bc} - \partial_{\mu_2}\mathring{A}^{a_2}_{\nu_2} + \partial_{\nu_2}\mathring{A}^{a_2}_{\mu_2} - \mathring{A}^{d_1}_{\mu_2}\mathring{A}^{e_1}_{\nu_2}C^{a_2}_{d_1e_1}\Big) \\ & + \lambda_2\ g^{\mu_2\mu_1}h^{a_2a_1}h_{b_1,b_2}\ \Big(\widehat{A}^{c_1}_{\mu_1}C^{b_1}_{c_1a_1} - \mathring{A}^{c_1}_{\mu_1}C^{b_1}_{c_1a_1}\Big)\cdot\Big(\widehat{A}^{c_2}_{\mu_2}C^{b_2}_{c_2a_2} - \mathring{A}^{c_2}_{\mu_2}C^{b_2}_{c_2a_2}\Big)\end{aligned}$$

Le champ τ n'est plus présent dans la théorie, la totalité de ses degrés de libertés ont été redistribué sur A_μ pour former le champ composite $\widehat{A}_\mu$. En tant que champs invariant sous l'action du groupe de jauge, les champs $\widehat{A}_\mu$ constituent des observables physiques. Le comptage des degrés de libertés de la théorie, avant et après la définition des champs composites, montrent que cette méthode repose simplement sur un changement de variables approprié dans l'espace fonctionnel des champs de la théorie. Au final, cette méthode a pour effet "d'éliminer" la représentation du groupe de jauge, le groupe de symétrie est toujours présent dans la théorie, il est toutefois neutralisé.

Cette neutralisation du groupe de symétrie s'accompagne de termes de masses pour les champs $\widehat{A}_\mu$. En effet, on peut lire dans le lagrangien précédent le terme de masse $m_{ab} = \frac{1}{mn}\ h^{a_1a_2}h_{b_1b_2}C^{b_1}_{aa_1}C^{b_2}_{ba_2}$. Finalement, cette méthode de "changement de variables" permet de passer d'une théorie de jauge décrivant la propagation de bosons de jauge A_μ, en interactions avec un champ scalaire tensoriel τ^b_a, à une théorie Yang-Mills décrivant la propagation de bosons vecteurs massifs $\widehat{A}_\mu$ et dont le groupe de symétrie n'agit plus sur les champs de la théorie.

Contents

Introduction

Domains of the modern physics can be separated into two distinct subsets accordingly to the nature of their objects. The first subset is composed of domains which are related to the study of natural phenomena, where mathematics play the role of nothing more than a "tool". The second subset is related to the study of physical objects which are deeply rooted in a mathematical framework. This subset is related to the discipline of mathematical physics. We consider that the mathematical physics as some sort of non-splittable mix between mathematical description and physical content.

Mathematics are present in domains of physics, from solid mechanics to general relativity. However, physics domains are not all concerned with mathematical physics. Indeed, it would be too wide to simply define mathematical physics as the discipline of the mathematical analysis of physical phenomena. Mathematics in physics should not be restricted to a powerful tool used to model physical systems and to solve equations. For example, solid mechanics, fluid mechanics and optical physics are concerned with, respectively, local forces applied to a continuous medium, interactions between fluid particles and optical properties of light. In these cases, the corresponding physical system exists independently of the formalism. At the opposite, quantum mechanics and general relativity are parts of mathematical physics in the sense that these domains cannot be understood without making references neither to vectors of an Hilbert space nor to the riemannian geometry, respectively.

It is important to determine what mathematical physics *is*, but also what it is *not*: mathematical physics is neither mathematics nor physics. Mathematics are related to the investigation of the properties of abstract objects and their classifications, whereas the purpose of physics is to establish predictive and repetitive laws concerning natural phenomena. But mathematical physics is neither concerned with the study of concrete problems or experiences, nor with the classification of mathematical objects and generalizations of results.

We consider that the mathematical framework carries a conceptual information concerning the meaning of physics. Then, the investigation and the generalization of the mathematics underlying a physical theory can lead to the prediction of new physical observations and/or measurable quantities. This assertion can be illustrated by some of the greatest results of the 20th century such as the prediction of anti-matter particles by Paul Dirac and the prediction of the gyro-magnetic factor of the electron using the Quantum ElectroDynamic formalism (these two examples are present in almost every textbook related to the quantum field theory, see [PS95; IZ85; Wei95]). Nevertheless, we do not ignore how many elegant mathematical models have been infirmed by experiences. The best example is the first unification theory between general relativity and electromagnetism by H. Weyl. Another one is the adding of a fifth dimension of spacetime in the Kaluza's theory. An historical description of these first attempts to the unification can be found in [OS99]. Facts and observations are final judges in physics, and theories find no legitimacy in the beauty of their mathematical construction.

Among many subjects of the mathematical physics, we are interested in gauge theories associated to groups of symmetry.

Since the special relativity of 1905, every objects have to be formulated under a covariant form to get the correct transformations with respect to the action of the Lorentz group. These objects are the famous quadri-vectors "x^μ", these Greek indices are related to their external degrees of freedom *i.e.* the degrees of freedom associated to their position in spacetime.

Besides, the standard model (SM) of particles physics considers the existence of symmetry groups as part of the fundamental ingredients in the description of the infinitesimal world. Here, each of these symmetry groups is associated to a fundamental force in physics: the groups of symmetry $U(1)$, $SU(2)$ and $SU(3)$ are associated to the electromagnetic interactions, weak interactions and strong interactions, respectively. They act on some representation spaces, *e.g.* vectors or tensorial fields, and affect only their inner degrees of freedom, usually denoted by Latin indices. These groups of symmetry are not related to some "external" displacements of particles in the spacetime, they do not affect the external degrees of freedom of the system. Then, any microscopical system can be equipped with both external degrees of freedom (related to its position in the spacetime) and internal degrees of freedom (related to its microscopic invariance with respect to a given interaction). Groups of inner symmetries do act on Latin indices, leaving invariant the Greek indices, whereas Lorentz group does act only on Greek indices.

This "duality" is illustrated by covariant derivatives used in particle physics. Initially, the covariant derivative was defined in order to add a minimal coupling between a scalar field ϕ and an external source A_μ. This could be done by substituting the derivative $\partial_\mu\phi$ (related to some "external" displacements of ϕ) with a covariant derivative $(\partial_\mu + A_\mu)\phi$ where A_μ is a matrix-valued field in the representation of ϕ. Then, the external infinitesimal displacement is "corrected" by action on the inner components of ϕ.

How can this duality be depicted in one global mathematical framework?

The theory of fiber bundles is related to the description of an upper space, defined "above" a manifold $\mathcal{M}$. The total space contains additional degrees of freedom so that it results in a scheme which "mixes" both the geometry of spacetime and some inner space. This inner space has no extension on the manifold, it can be seen as a space "perpendicular" to $\mathcal{M}$. Then, it should be considered as a set of new degrees of freedom which cannot be directly "seen" by physics. For principal bundles, a Lie group G moves points of $\mathcal{P}$ vertically, along their fibers. This corresponds to the so-called inner group of symmetry. Here, external and inner degrees of freedom are present in one global mathematical framework. The external degrees of freedom will be refereed to as the *geometric* degrees of freedom.

In 1954, C. N. Yang and R. L. Mills built the so-called Yang-Mills (YM) theories which generalize the Maxwell $U(1)$-theory of electromagnetism to higher groups of symmetry (they are isotopic groups in [YM54]). The construction of the YM models consists in taking into account, for any fields which are in a representation of a group of symmetry, that infinitesimal displacements on spacetime are related to some an external and inner infinitesimal displacement. In the formalism of fiber bundles, this leads to equip the principal bundle $\mathcal{P}$ with a *connection*.

The role of connections on $\mathcal{P}$ are similar to Christoffel symbols in general relativity. On a curved Riemannian manifold, the usual "flat" derivative is corrected by Christoffel

	Scalar fields in YM models	Infinitesimal variations
External degrees of freedom	$\phi = \phi(x^1, \ldots, x^m)$	$\mathrm{d}\phi = \partial_\mu \phi(x^1, \ldots, x^m) \otimes \mathrm{d}x^\mu$
Inner degrees of freedom	$\phi = \phi^a \otimes E_a$	$A{\cdot}\phi = A^a_b \phi^b \otimes E_a$

Table 1: Infinitesimal variations of scalar fields in both external and inner directions.

symbols to take into account the curvature of the manifold. On a principal bundle $\mathcal{P}$, connections correct the "trajectories" of fields defined on spacetime, taking into account these inner degrees of freedom. It results in the definition of covariant derivatives. Local trivializations of covariant derivatives are symbolically denoted as $(\mathrm{d} + A)$ where d is related to external infinitesimal displacements (related by the partial derivative ∂_μ) and $A = A^a_\mu \mathrm{d}x^\mu \otimes \mathfrak{g}$ acts on the inner components of the fields. See Table 1.

The components A^a_μ come from the local trivialization of the connection 1-form ω defined on $\mathcal{P}$. In gauge field theories of the particle physics, these correspond to the so-called gauge bosons, mediators of the interaction associated to this group. Connections also define geometric curvatures, and YM theories are simply obtained by the "norm" of curvatures. This norm corresponds to the kinetic term in particle physics.

What is the role of gauge theories in this geometrical context?

Gauge theories are theories which involves fields defined on a manifold $\mathcal{M}$ equipped with both inner and external degrees of freedom and which support a representation of a symmetry group, called the *gauge group*, which acts only on the inner components of these fields. Then, these so-called *gauge fields* form a multiplet with respect to the symmetry group. As a gauge principle, we claim that physical observables have to be gauge-invariant quantities *i.e.* every "inner configurations" of the multiplet which are related by the action of the gauge group are equivalent with respect to the physical observation. According to this gauge principle, gauge fields whose components are not invariant with respect to the action of a gauge group are not observables.

We stress the distinction between the so-called *active* and *passive* gauge transformations. The former corresponds to an action of the gauge group on the inner components of the fields of a gauge theory. If the gauge theory is associated to a principal bundle, active transformations are encoded into vertical automorphisms of principal bundle. By moving points along their fibers, objects defined on $\mathcal{P}$ undergo some geometric transformations. The latter corresponds to transformations which occur by changes of local trivializations of principal bundle *e.g.* from an open set $\mathcal{U} \subset \mathcal{M}$ to another. With respect to the passive gauge transformations, the gauge principle states that observables in physics have to be independent of the choice of trivialization of $\mathcal{P}$. In general case, mathematical expressions of active and passive transformations are the same. These two expressions will be distinguished in chapter 6.

How are transitive Lie algebroids related to gauge theories?

The theory of Lie algebroids was originally defined as the infinitesimal version of the Lie groupoids (the full aspect of the theory of Lie groupoids and Lie algebroids is presented in

[Mac05], a quick survey of the theory can be found in [Wei96]). They were first introduced by J. Pradines in 1967 [Pra67]. A Lie algebroid is a vector bundle $\mathcal{A}$ defined over a manifold $\mathcal{M}$, equipped a $C^\infty(\mathcal{M})$-linear pointwise map $\rho : \mathcal{A} \to T\mathcal{M}$, called the anchor, which projects elements of $\mathcal{A}$ to elements of the tangent bundle $T\mathcal{M}$, and a Lie bracket defined on the space of sections $\mathsf{A} = \Gamma(\mathcal{A})$ such that, for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$ and $f \in C^\infty(\mathcal{M})$, one has

$$[\mathfrak{X}, f{\cdot}\mathfrak{Y}] = f{\cdot}[\mathfrak{X}, \mathfrak{Y}] + (\rho(\mathfrak{X}){\cdot}f){\cdot}\mathfrak{Y}$$

If the anchor ρ is surjective, $\mathcal{A}$ is said to be a transitive Lie algebroid. The kernel of ρ is denoted by $\mathcal{L}$ and we denote the space of sections of this vector bundle as $\mathsf{L} = \Gamma(\mathcal{L})$. Lie algebroids are usually used in connection of mechanics when they are mostly described in terms of fiber bundles (for a review concerning the relations and applications to the Poisson mechanics, see [KS08] and references therein)

Here, in order to get closer to the formalism of gauge theories, we adopt the sectional point of view of $\mathcal{A}$. Then, we consider the finite projective $C^\infty(\mathcal{M})$-module A as the Lie algebroid so that elements of A are maps $\mathfrak{X} : \mathcal{M} \to \mathcal{A}$. This description is similar to the formalism used in the noncommutative geometry (NCG) of A. Connes (we refer the reader to the standard book [Con94; CM08a]). Transitive Lie algeboids are considered as a generalization of the tangent bundle $T\mathcal{M}$. Obviously, for $\rho = \mathrm{Id}$, we obtain $\mathcal{A} = T\mathcal{M}$, and then the Lie algebroid is exactly the tangent bundle of $\mathcal{M}$. In our description, the anchor ρ projects only a "part" of $\mathcal{A}$ to elements of the tangent bundle. In this sense, the complementary part represents a generalization of $T\mathcal{M}$. From the point of view of sections, transitive Lie algebroids are generalizations of *vector fields* on $\mathcal{M}$. The part of A which projects to $\Gamma(T\mathcal{M})$ is called the *geometric component* of A and the kernel of the anchor is called the *algebraic component* of A.

We are interested in defining global gauge field theories based on transitive Lie algebroids. We expect to generalize the YM type models defined in differential geometry. Then, the study of new terms, specific to the formalism of transitive Lie algebroids could possibly lead to new theoretical predictions and/or observations.

The first step consists into exploring the structures of transitive Lie algebroids to correctly understand how they extend objects and constructions related to vector fields on $\mathcal{M}$. The extended part of transitive Lie algebroids is explicit on trivial Lie algebroids, which are written as the direct sum of the space of vector fields on $\mathcal{M}$ and the space of functions on $\mathcal{M}$ with values in $\mathfrak{g}$ *i.e.* as where $\mathsf{A} = \mathsf{TLA}(\mathcal{M}, \mathfrak{g}) = \Gamma(T\mathcal{M}) \oplus \Gamma(\mathcal{M} \times \mathfrak{g})$. In non-trivial situations, these two spaces are "melted" in A. Local trivializations of A are defined by using local isomorphisms of Lie algebroids from A to trivial Lie algebroids $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ on $\mathcal{U} \subset \mathcal{M}$ modeled over a Lie algebra $\mathfrak{g}$.

The formalism in terms of sections is compatible with the the definition of differential complexes on A. Two differential complexes are relevant: the graded algebra of differential forms on A with values in $C^\infty(\mathcal{M})$, and the graded Lie algebra of differential forms on A with values in L. The former has already been studied by several authors as a generalization of the de Rham differential complex (notable results related to this space are given by K. Kubarski in [Kub98; Kub99; KM03]). The latter plays here a significant role in constructions of gauge invariant theories. By using local trivializations of transitive Lie algebroids, differential complexes defined on A are locally isomorph to differential complexes defined on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. These would correspond to the "local version" of the "global" differential forms.

On transitive Lie algebroids, we define the space of *ordinary connections*, in terms of a differential form defined on A with values in L. On Atiyah Lie algebroids, the space of ordinary connections 1-forms are in $1:1$ correspondence with the Ehresmann connections on a principal fiber bundle $\mathcal{P}$ (see the works of S. Lazzarini and T. Masson in [LM12a]). Ordinary connections are geometrical objects in the sense that they do not see the algebraic component of A. This is transcripted into the relation $\omega \circ \iota(\ell) = -\ell$ for any $\ell \in \mathsf{L}$, this corresponds to the *normalization* of ω on L. By relaxing this constraint, ordinary connections are easily extended to the space of *generalized connections* on A.

Given a background connection, we show that any generalized connections can be explicitly written as the sum of an ordinary connection and a purely algebraic term, the *reduced kernel endomorphism* $\tau : \mathsf{L} \to \mathsf{L}$, which does not exist in the differential geometry approach. This algebraic term goes with the induced ordinary connection throughout the following constructions.

Finally, transitive Lie algebroids support an infinitesimal gauge action defined by a representation of the kernel L. In the case of Atiyah Lie algebroids, this representation of L is exactly the infinitesimal version of the action of the gauge group. On the space of generalized connections, the kernel L can be represented either by the Cartan operation associated to L, or by an algebraic operation, compatible with the covariant derivative associated to ϖ.

What is the outcome of all this?

All these ingredients are implemented into a gauge theory constructed using a specific metric on A, a Hodge star operator and an integral operator over A. It results in the definition of a Lagrangian density $\mathcal{L}[A, \tau]$ which depends only on the geometric local connection $A = A^a_\mu \mathrm{d}x^\mu \otimes E_a$, and on some scalar fields τ^b_a, coupled to the gauge bosons A, embedded into a potential term. Obviously, this theory is related to the well-known Yang-Mills-Higgs (YMH) models associated to Brout-Englert-Higgs-Hagen-Guralnik-Kibble (BEHHGK, pronounced “beck”) mechanism of spontaneous symmetry breaking (also known as the Higgs mechanism, see [PS95; IZ85] for the full theory of spontaneous symmetry breaking, including the quantization of the associated theory). This mechanism is an essential feature to get massive vector bosons.

This gauge theory is the main result of this PhD thesis. It proves that transitive Lie algebroids equipped with generalized connections contain scalar fields, as algebraic parameters, not present in differential geometry, whose role is similar to the scalar field in BEHHGK mechanism. By putting this element to zero, generalized connections become an ordinary connection 1-forms, and the gauge field theory based on transitive Lie algebroids becomes exactly a YM theory.

Scalar fields are not introduced by hand in the theory, as for the BEHHGK mechanism. From the mathematical point of view, this construction stays in the *geometry of the (generalized) connections* on A. Analogies and differences between YMH models based on generalized connections and YMH models based on the BEHHGK mechanism will be discussed in the body of the text.

In particular, in the BEHHGK mechanism, the scalar field ϕ is embedded into a potential term with possibly non-zero minimum, called the vacuum configurations. The “shape” of this potential term depends on dynamical parameters related to the energy scale of the system. In analogy with ferromagnetic materials, the field ϕ can be spontaneously polarized in one direction. Then, the initial symmetry group is reduced to one of its subgroup in order to preserve the polarization of ϕ. In YMH models based on transitive Lie algebroids, the free parameters of the theory do come from the formulation of A. They do not correspond to dynamical parameters. Then, the "shape" of the potential is invariant and is simply written as an algebraic constraint on τ.

There exists several methods of symmetry reduction. In addition of the BEHHGK mechanism, the symmetry group can be reduced using *gauge-fixing term* or *reduction of fiber bundles* (This last point is a mathematical result of the differential geometry, see [KN96a]). Here, procedures of symmetry reduction are substituted by a new method of "neutralization" of the symmetry group. To this purpose, we restrict the theory to a suitable subspace of generalized connections. Then, we show that degrees of freedom of τ can be "moved" to the gauge boson A_μ in order to form a gauge invariant composite field $\widehat{A}_\mu$. It results in a theory where the representation of the gauge group is trivial on fields of the theory. Moreover, gauge invariant composite fields $\widehat{A}_\mu$ acquire mass terms. Under its final form, the theory describes a YM type theory with massive vector fields.

I now announce the plan of my PhD thesis.

In chapter 1, we expose the main definitions of differential geometry and theory of fiber bundles. This topic is far too wide to be fully detailed in a PhD thesis and we refer the reader to [KN96a; KN96b; Ste60] for mathematical descriptions of differential geometry and to [TS87; Nak03; CN87; Fel87] for applications to physics. Instead, we establish the usual gauge construction into a more algebraic description. This description will be in correspondence with the formalism of the transitive Lie algebroids.

In chapter 2, we give the general theory of the (transitive) Lie algebroids, mainly in terms of sections defined on $\mathcal{M}$ with values in $\mathcal{A}$. Actually, Lie algebroids are usually describe in terms of fiber bundles and are strongly related to Poisson mechanics. Here, we adopt the point of view of the gauge field theory and the NCG. We insist on the coexistence of both the geometric and the algebraic degrees of freedom of A. This point becomes explicit in the case of trivial Lie algebroids. We also illustrate transitive Lie algebroids by defining Atiyah Lie algebroids associated to a principal bundle $\mathcal{P}$ (see [Ati57]). Local trivializations of transitive Lie algebroids are defined in terms of isomorphisms of Lie algebroids between generic transitive Lie algebroids A and trivial Lie algebroids defined on $\mathcal{U} \subset \mathcal{M}$.

In chapter 3, we define graded differential algebras of forms defined on A with values in a representation space $\mathcal{E}$ equipped with differential operators associated to their corresponding representation of A. These objects generalize differential structures defined on $\mathcal{M}$ (Classical results of algebraic topology can be found in [BT10; ES05]). Examples of graded differential algebras are given by $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M})$ and $\Gamma(\mathcal{E}) = \mathsf{L}$. It results in the definition of the differential complexes $(\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}})$ and $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$. Locally, these two differential complexes are isomorph to the differential complexes $(\Omega_{\mathsf{TLA}}(\mathcal{U}), \delta)$ and $(\Omega_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$ (their "local" versions), respectively.

Chapter 4 gives the general theory of ordinary and generalized connections on transitive Lie algebroids. The notion of ordinary connections defined on transitive Lie algebroid is close to the notion of ordinary connections defined in NCG (see [MS05]). Ordinary connections "inject" the geometry of vector fields into the Lie algebroid A. They are equivalently defined by differential 1-forms in $\Omega^1(\mathsf{A}, \mathsf{L})$ which are normalized on L. Ordinary connections are also related the definition of some "horizontal" subspace in A. By relaxing the normalization on L, we define the space of generalized connections on transitive Lie algebroids. Given a background connection, any generalized connection ϖ can be decomposed into an ordinary connection 1-form and a reduced kernel endomorphism

τ associated to ϖ. For both ordinary and generalized connections, covariant derivatives and curvatures are well-defined.

In chapter 5, we give all the necessary tools in order to define a scalar product on differential of forms on A. To do so, we define some metrics $\widehat{g}$ on A, an inner integral operator acting on $\Omega^\bullet(\mathsf{A})$, an integral over A and we generalize the Hodge star product to transitive Lie algebroids in order to take into account both the geometric and the algebraic degrees of forms of $\Omega^\bullet(\mathsf{A})$. Finally, we obtain a scalar product $\langle\cdot,\cdot\rangle : \Omega^\bullet(\mathsf{A}) \times \Omega^\bullet(\mathsf{A}) \to \mathbb{R}$. With respect to this scalar product, differential forms of distinct degrees are orthogonal.

To construct gauge invariant theories based on transitive Lie algebroids, we define in chapter 6 the infinitesimal gauge action of L on both ordinary and generalized connections on A and on their associated objects *i.e.* covariant derivatives and curvatures. In particular, on the space of generalized connections on A, the infinitesimal action of L can be defined either as the Lie derivative associated to the Cartan operation $(\mathrm{L}, i, \mathcal{L})$ or as a new "algebraic" action of L. With respect to the latter, covariant derivatives and curvatures, associated to generalized connections, have homogeneous gauge transformations.

Chapter 7 is devoted to the computation of the gauge invariant action functional, defined as the "norm" of the curvature associated to any generalized connections on A. The Lagrangian associated to this action describes a YMH type theory where the usual scalar field of the SM is substituted with scalar fields τ_a^b, embedded into an algebraic potential term. Some mathematical and physical discussions will be present to illustrate the relevance of the obtained result.

Chapter 8 is an application of a general method of symmetry reduction based on a change of variables in the functional space of the fields of the theory (full theory detailed in [FFLM13]). Application to gauge theories on Atiyah Lie algebroids gives a description of massive vector fields.

Chapter 1

Differential geometry

This chapter is devoted to the basic definitions, notations, and constructions of both the differential geometry and the theory of fiber bundles. Although this formalism is well-known for most of the theoretical physicists (see [KN96a; KN96b; Nak03; TS87; CN87]), we will rather focus on a description in terms of algebraic objects defined as modules over $C^\infty(\mathcal{M})$ than in terms of geometric bundles, dual bundles, etc. This approach is close to the formalism of the NCG which consists into describing geometric objects in terms of module structures (see [Con94; CM08a]). Moreover, this algebraic language is well-adapted to the constructions of gauge field theories based on $\mathcal{M}$.

In chapter 2, transitive Lie algebroids are considered as a generalization of the space of vector fields on $\mathcal{M}$. Then, many constructions presented here will be generalized to this new framework. The "algebraic" formalism we will use in this chapter will permit to make comparisons between corresponding objects.

1.1 Differentiable manifolds

1.1.1 Manifolds

A manifold is a topological space which can be seen locally as a "flat" surface *e.g.* euclidean or minkowskian. However, globally, this flatness is lost to the benefit of some intrinsic curvature.

Differentiable manifolds, or simply *manifolds*, are topological spaces $\mathcal{M}$ which are covered by a finite number of open sets $(\mathcal{U}_i)_{\mathrm{i}\in I}$, with $\underset{i\in I}{\cup} \mathcal{U}_i = \mathcal{M}$, such that, for each open set $\mathcal{U}_i$, there exists an invertible homeomorphism $\varphi_i : \mathcal{U}_i \to \mathcal{O}$ where $\mathcal{O}$ is an open set of $\mathbb{R}^m$ where m is the dimension of the manifold. Thus, manifolds are considered as a collection of "flat" spaces, glued together to form the total space.

Each pair $(\mathcal{U}_i, \varphi_i)$ represents a chart of $\mathcal{M}$. The union of all the charts of $\mathcal{M}$ forms an *atlas* of $\mathcal{M}$. Each chart permits to describe the neighborhood of any point p in terms of coordinates in $\mathbb{R}^m$. We denote by $(e_1, e_2, ..., e_m)$ a basis of $\mathbb{R}^m$. With respect to this basis, any element $p \in \mathcal{U}_i$ can be locally described in $\mathcal{O} \in \mathbb{R}^m$ as $\varphi_i(p) = x_i^1 e_1 + x_i^2 e_2 + \ldots + x_i^m e_m = x_i^\mu e_\mu$ where $x_i^\mu \in \mathbb{R}$ are the *coordinates* of p.

On $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$,[1] the map $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$ is an invertible homeomorphisms from $\varphi_j(\mathcal{U}_i \cap \mathcal{U}_j)$ to $\varphi_i(\mathcal{U}_i \cap \mathcal{U}_j)$. [2] If every $\mathcal{U}_{ij}$ are diffeomorphic to $\mathbb{R}^m$, the cover $(\mathcal{U}_i)_{i\in I}$ is said

[1] Through all this PhD thesis, we denote by $\mathcal{U}_{ij}$ the non-empty intersection of the open sets $\mathcal{U}_i$ and $\mathcal{U}_j$. We respect the Čech convention $\mathcal{U}_{ij} = -\mathcal{U}_{ji}$.

[2] Note that the index of the target space is located at the left of the index of the source space. This convention will be adopted through all this PhD thesis (except in subsection 2.2.2, where we use a vertical disposition of the indices).

to be *good cover (of finite type)*.[3] The set of homeomorphisms φ_{ij}, for $i, j \in I$, fulfills the structure conditions

$$\begin{cases} \varphi_{ji} = \varphi_{ij}^{-1} \\ \varphi_{ii} = \mathrm{Id}_{\varphi_i(\mathcal{U}_i)} \\ \text{On } \mathcal{U}_{ijk} = \mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k, \text{ one has } \varphi_{kj} \circ \varphi_{ji} = \varphi_{ki} \end{cases} \tag{1.1.1}$$

We denote by $C^\infty(\mathcal{M})$ the space of functions $f : \mathcal{M} \to \mathbb{R}$ of class C^∞. [4] Locally, with respect to a chart $(\mathcal{U}_i, \varphi_i)$ of $\mathcal{M}$, we use the same letter to define both the maps $f : \mathcal{M} \to \mathbb{R}$ and $f : \varphi_i(\mathcal{U}_i) \to \mathbb{R}$ so that we have $f(p) = f(x_i^1, x_i^2, \ldots, x_i^m)$ for any $p \in \mathcal{U}_i$. This will not lead to ambiguous notations. Let $f, g \in C^\infty(\mathcal{M})$, the product $f{\cdot}g$ is defined as $(f{\cdot}g)(p) = f(p)g(p)$ for any $p \in \mathcal{M}$.

On the open set $\mathcal{U}_{ij}$, any point p can be either written in the coordinates $(y^1, y^2, \ldots, y^m)$, with respect to the chart $(\mathcal{U}_j, \varphi_j)$, or in the coordinates $(x^1, x^2, \ldots, x^m)$, with respect to the chart $(\mathcal{U}_i, \varphi_i)$. The determinant of the Jacobian matrix $J_{ij} \in C^\infty(\mathcal{M})$ is defined, at $p \in \mathcal{U}_{ij}$, as

$$J_{ij}(p) = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^m} \\ \vdots & & \vdots \\ \frac{\partial y^m}{\partial x^1} & \cdots & \frac{\partial y^m}{\partial x^m} \end{vmatrix} (p) \tag{1.1.2}$$

A manifold $\mathcal{M}$ is said to be *orientable* if there exists an atlas of $\mathcal{M}$ which fulfills the condition $J_{ij}(p) > 0$ for any $i, j \in I$.

1.1.2 Vector fields on $\mathcal{M}$

On manifolds, *tangent vectors*, or simply *vectors*, are defined as the "derivations", at one point p, of an equivalence class of curves passing through this point. Let $\Phi_t(p)$ be a curve on $\mathcal{M}$ parametrized by an index $t \in [-1; 1]$ such that $\Phi_0(p) = p$. The tangent vector X_p, associated to this curve, is defined at the point p as:

$$X_p : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}) \quad ; \quad (X_p{\cdot}f)(p) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\Phi_t(p)) \tag{1.1.3}$$

The set of all the tangent vectors at the point p is a vector space of rank m. It is denoted by $T_p\mathcal{M}$ and is called the *tangent space* of $\mathcal{M}$ at the point p. The union of every tangent spaces $T_p\mathcal{M}$ forms the *tangent vector bundle*, or *tangent bundle*, and is denoted by $T\mathcal{M} = \underset{p\in\mathcal{M}}{\cup} T_p\mathcal{M}$.

The space of vector fields are defined as the space of *sections* of $T\mathcal{M}$ *i.e.* by the set of maps $X : \mathcal{M} \to T\mathcal{M}$, defined as $X : p \mapsto X_p \in T_p\mathcal{M}$. Vector fields can also be seen as a *smooth* assignment of a vector to each point of $\mathcal{M}$. The *flow* of a vector field X is a parametrized map $\Phi_X : \mathcal{M} \times \mathbb{R} \to \mathcal{M}$ defined at $p \in \mathcal{M}$ as:

$$X_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Phi_{X,t}(p) \quad \text{and} \quad \Phi_{X,0}(p) = p \tag{1.1.4}$$

for any $p \in \mathcal{M}$. Vector fields acts on $C^\infty(\mathcal{M})$ as $X{\cdot}f = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \phi^*_{X,t} f$, for any $f \in C^\infty(\mathcal{M})$.

[3] This will be the case in all this PhD thesis.

[4] Every functions defined on manifolds will be of class C^∞.

With respect to a chart $(\mathcal{U}_i, \varphi_i)$, we denote by $(\partial_{i,1}, \partial_{i,2}, \ldots, \partial_{i,m})$ a basis of the tangent space at $p \in \mathcal{U}_i$. Then, locally, any vector field $X \in \Gamma(T\mathcal{U}_i)$ can be written as $X = X^\mu \partial_{i,\mu}$ where $X^\mu \in C^\infty(\mathcal{U}_i)$ are the *components* of X, so that $X \cdot f = X^\mu \partial_{i,\mu} f$ where $\partial_{i,k}$ stands for $\partial/\partial x_i^k$. On $\mathcal{U}_{ij}$, the elements $(\partial/\partial x^\mu)$ are related to elements $(\partial/\partial y^\mu)$,by changes of charts, as

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu}. \tag{1.1.5}$$

Then, on the open subset $\mathcal{U}_{ij}$, any tangent vector X_p can be written either as $X_p = X_p^\mu (\partial/\partial x^\mu)$, with respect to the chart $(\mathcal{U}_i, \varphi_i)$, or as $X_p = Y_p^\mu (\partial/\partial y^\mu)$, with respect to the chart $(\mathcal{U}_j, \varphi_j)$. Then, the components Y_p^μ and X_p^μ are related as $Y_p^\mu = \left(\frac{\partial y^\mu}{\partial x^\nu}\right)(p)\, X_p^\nu$.

The space $\Gamma(T\mathcal{M})$ is equipped with a $C^\infty(\mathcal{M})$-linear antisymmetric Lie bracket $[\cdot,\cdot] : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$. It is defined, for any $f \in C^\infty(\mathcal{M})$, $X, Y \in \Gamma(\mathcal{M})$, as the commutator

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f). \tag{1.1.6}$$

Equivalently, we define the Lie bracket on $\Gamma(T\mathcal{M})$ in terms of the flows. Let $\Phi_{X,t}(p)$ and $\Phi_{Y,s}(p)$ be the flows associated to the vectors $X, Y \in T_p\mathcal{M}$, respectively. Then, the Lie bracket is defined as:

$$[X, Y]_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left(\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \left(\Phi_{X,-t} \circ \Phi_{Y,s} \circ \Phi_{X,t}(p) \right) \right). \tag{1.1.7}$$

1.1.3 Differential forms on $\mathcal{M}$

From the geometric point of view, *differential forms* of degree q are sections of the fiber bundle $\wedge^q(T^*\mathcal{M})$, where $T^*\mathcal{M}$ denotes the cotangent bundle over $\mathcal{M}$ and $\wedge^q$ denotes the multi-linear antisymmetric tensorial product of q vector spaces. By definition, one has the equality

$$\Gamma(\wedge^q(T^*\mathcal{M}) = \wedge^q \Gamma(T^*\mathcal{M}) \tag{1.1.8}$$

Here, we rather use an "algebraic" definition of differential forms than a description in terms of sections of cotangent bundles. We define the space of differential forms of degree q as the vector space of the $C^\infty(\mathcal{M})$-multilinear antisymmetric maps : $\wedge^q \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$. This space is a module over $C^\infty(\mathcal{M})$ and we denote it by $\Omega^q(\mathcal{M})$. A differential q-form ω is completely antisymmetric in the sense that

$$\omega(X_1, \ldots, X_i, \ldots, X_j, \ldots, X_q) = -\omega(X_1, \ldots, X_j, \ldots, X_i, \ldots, X_q) \tag{1.1.9}$$

for any $X_1, \ldots, X_q \in \Gamma(T\mathcal{M})$ and $i, j = 1, \ldots, n$. Geometrically, the space $\Omega^1(\mathcal{M})$ corresponds to the space of covectors on $\mathcal{M}$ *i.e.* the space of sections on the cotangent bundle $T^*\mathcal{M}$.

With respect to the chart $(\mathcal{U}_i, \varphi_i)$, we denote by $(\mathrm{d}x_i^1, \mathrm{d}x_i^2, \ldots, \mathrm{d}x_i^m)$ a basis of $T^*\mathcal{U}_i$. Any element $\mathrm{d}x_i^\mu$ is dual to the element $\partial_{i,\nu}$ in the sense that $\mathrm{d}x_i^\mu(\partial_{i,\nu}) = \delta_\nu^\mu$ where δ_μ^ν is the Kronecker symbol. Then, any 1-form $\omega \in \Omega^1(\mathcal{M})$ can be written as $\omega = \omega_\mu \mathrm{d}x^\mu$ where $\omega_\mu \in C^\infty(\mathcal{M})$ are the components of ω, so that $\omega(X) = \omega_\mu X^\mu$, for any $X \in \Gamma(T\mathcal{M})$, and any differential q-form ω can be written as

$$\omega = \frac{1}{q!} \omega_{\mu_1 \mu_2 \ldots \mu_q} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_q} \tag{1.1.10}$$

where $\omega_{\mu_1 \mu_2 \ldots \mu_n} \in C^\infty(\mathcal{U}_i)$ is completely antisymmetric. Then, one has

$$\omega(X_1, \ldots, X_q) = \omega_{\mu_1 \mu_2 \ldots \mu_q} X^{\mu_1} \ldots X^{\mu_q} \tag{1.1.11}$$

From the antisymmetric property of the q-form ω, it vanishes if $q > m$ where m is the dimension of $\mathcal{M}$.

We use the multiplication in $C^\infty(\mathcal{M})$ to define a graded tensorial product on $\Omega^q(\mathcal{M})$. Let $\omega \in \Omega^q(\mathcal{M})$ and $\eta \in \Omega^r(\mathcal{M})$, then $\omega \wedge \eta$ is a $(q+r)$-form defined on $\mathcal{M}$ with values in $C^\infty(\mathcal{M})$, such that $\omega \wedge \eta = (-1)^{q+r} \eta \wedge \omega$, and defined as

$$(\omega \wedge \eta)(X_1, \dots, X_{q+r}) = \frac{1}{q!r!} \epsilon^{a^1 a^2 \dots a^{q+r}} \omega(X_{a^1}, \dots, X_{a^q}) \cdot \eta(X_{a^{q+1}}, \dots, X_{a^{q+r}}) \quad (1.1.12)$$

for any $X_1, \dots, X_{q+r} \in \Gamma(T\mathcal{M})$ and $\epsilon^{a^1 a^2 \dots a^{q+r}}$ is the completely antisymmetric Levi-Civita tensor. On $\mathcal{U}_{ij}$, differential q-forms ω can be locally written either accordingly to the basis $(\mathrm{d}x^1, \dots, \mathrm{d}x^q)$, with respect ot the chart $(\mathcal{U}_i, \varphi_i)$, or accordingly to the basis $(\mathrm{d}y^1, \dots, \mathrm{d}y^q)$, with respect to the chart $(\mathcal{U}_j, \varphi_j)$. With respect to these two decompositions, the components $\omega_{i,\, \mu_1 \dots \mu_q}$ and $\omega_{j,\, \mu_1 \dots \mu_q}$ are related as

$$\omega_{j,\, \mu_1 \dots \mu_q} = \frac{\partial x^{\nu_1}}{\partial y^{\mu_1}} \dots \frac{\partial x^{\nu_q}}{\partial y^{\mu_q}} \, \omega_{i,\, \nu_1 \dots \nu_q}. \quad (1.1.13)$$

The graded differential complex $(\Omega^\bullet(\mathcal{M}), \mathrm{d})$ of differential forms defined on $\mathcal{M}$ with values in $C^\infty(\mathcal{M})$, is defined as the graded space $\Omega^\bullet(\mathcal{M}) = \oplus_{k=0} \Omega^k(\mathcal{M})$, where $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$, equipped with a differential operator d which increases the degree of forms by 1. This differential is the *Koszul derivative*, defined by using the *representation* of vector fields on $C^\infty(\mathcal{M})$ and also the Lie bracket on $\Gamma(T\mathcal{M})$. Let ω be a q-form, then $\mathrm{d}\omega$ is a $(n+1)$-form obtained from the *Koszul* formula as

$$\begin{aligned}(\mathrm{d}\omega)(X_1, \dots, X_{q+1}) = & \sum_i^{q+1} (-1)^{i+1} X_i \cdot \omega(X_1, \dots, \overset{i}{\vee}, \dots, X_{q+1}) \\ & + \sum_{1 \le i < j \le q+1} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1})\end{aligned}$$

for any $X_1, \dots, X_{q+1} \in \Gamma(T\mathcal{M})$ and where the symbols $\overset{i}{\vee}$ and $\overset{j}{\vee}$ denote the missing indices. The Koszul derivative is *nilpotent* is the sense that $(\mathrm{d} \circ \mathrm{d})\omega = 0$ for any n-form ω. In chapter 2, differential forms defined on transitive Lie algebroids preserve the same structure *e.g.* a space of module, a Lie bracket defined on the source space and a representation.

1.1.4 Integration over $\mathcal{M}$

A *volume form* on $\mathcal{M}$ is a differential m-form $\omega_{\mathrm{vol}} \in \Omega^m(\mathcal{M})$ which vanishes nowhere on $\mathcal{M}$. Such a form exists if and only if $\mathcal{M}$ is orientable. With respect to the local chart $(\mathcal{U}_i, \varphi_i)$, this volume form can be locally decomposed as $\omega_{i,\, 12 \dots m} \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^m$ where $\omega_{i,\, 12 \dots m} \in C^\infty(\mathcal{U}_i)$. The *integration* of a volume form over an open set $\mathcal{U}_i$, is given by the formula:

$$\int_{\mathcal{U}_i} \omega = \int_{\varphi_i(\mathcal{U}_i)} \omega_{i, 12 \dots m} \mathrm{d}x^1 \mathrm{d}x^2 \dots \mathrm{d}x^m \in \mathbb{R} \quad (1.1.14)$$

Let $(\rho_i)_{i \in I} : \mathcal{U}_i \to \mathbb{R}$ be a partition function of the unity. The m-form $\rho_i \omega$ is defined on $\mathcal{U}_i$ and is 0 elsewhere, for any $i \in I$. The integration of a volume form over $\mathcal{M}$ is given by the formula:

$$\int_{\mathcal{M}} \omega = \sum_{i \in I} \int_{\varphi_i(\mathcal{U}_i)} (\rho_i \omega) \quad (1.1.15)$$

1.1.5 Metric on $\mathcal{M}$

A metric g on $\mathcal{M}$ is a bilinear symmetric map $g : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$. It is *non-degenerate* if $g(X, Y) = 0$ for any $Y \in \Gamma(T\mathcal{M})$ if and only if $X = 0$. A metric g on $\mathcal{M}$ is *positive-definite* if $g(X, X) > 0$ for any $X \in \Gamma(T\mathcal{M})$.

In a chart $(\mathcal{U}_i, \varphi_i)$, the metric g can be written in terms of its components as $g_{\mu\nu} = g(\partial_\mu, \partial_\nu)$, where $(\partial_1, \dots, \partial_m)$ denotes a basis of $\Gamma(T\mathcal{M})$. Over $\mathcal{U}_{ij}$, the metric g can be either written as $g_{i,\,\mu\nu}$ with respect to the chart $(\mathcal{U}_i, \varphi_i)$ or as $g_{j,\,\mu\nu}$ with respect to the chart $(\mathcal{U}_i, \varphi_i)$. These two components are related by the formula $g_{i,\,\mu\nu} = \frac{\partial y^\rho}{\partial x^\mu}\frac{\partial y^\sigma}{\partial x^\nu} g_{j,\,\rho\sigma}$

1.1.6 Hodge Isomorphism

Let $\mathcal{M}$ be an orientable manifold equipped with a metric g. The *Hodge star operator* $\star : \Omega^q(\mathcal{M}) \to \Omega^{m-q}(\mathcal{M})$ realizes an isomorphism of vectorial spaces between differential forms of degree p and differential forms of degree $(m - p)$. The inverse of the Hodge star operation is defined by the relation $\star\star\omega = (-1)^{q(m-q)}\omega$, for any $\omega \in \Omega^q(\mathcal{M})$.

We denote by $(\mathrm{d}x_i^1, \mathrm{d}x_i^2, \dots, \mathrm{d}x_i^m)$ a basis of the cotangent bundle $T^*\mathcal{U}_i$. To obtain an explicit expression of the Hodge star operator, we decompose locally the q-form $\omega \in \Omega^q(\mathcal{M})$, with respect to the chart $(\mathcal{U}_i, \varphi_i)$, as $\omega_i = \frac{1}{q!}\omega_{i,\,\mu_1\mu_2\dots\mu_q}\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x_i^{\mu_2} \wedge \dots \wedge \mathrm{d}x_i^{\mu_q}$ where $\omega_{i,\,\mu_1\mu_2\dots\mu_q} \in C^\infty(\mathcal{U}_i)$. Then, the Hodge star operator acts on ω_i as

$$\star\omega_i = \frac{1}{(m-q)!}\omega_{i,\,\mu_1\mu_2\dots\mu_q}\epsilon_{\nu_1\nu_2\dots\nu_m}\delta^{\mu_1\nu_1}\delta^{\mu_2\nu_2}\dots\delta^{\mu_q\nu_q}\mathrm{d}x^{\nu_{q+1}} \wedge \mathrm{d}x^{\nu_{q+2}} \wedge \dots \wedge \mathrm{d}x^{\nu_m} \quad (1.1.16)$$

where $\epsilon_{\nu_1\nu_2\dots\nu_m}$ is the completely antisymmetric Levi-Civita tensor with $\epsilon_{12\dots m} = 1$ and δ^{ab} is the Kronecker symbol. Here, $\star\omega_i$ is a $(m - q)$-form defined on $\mathcal{U}_i$. Over $\mathcal{U}_{ij}$, by changes of charts, we obtain $\star\omega_i = \star\omega_j$, for any $i, j \in I$, so that the $(m - q)$-form $\star\omega$ is globally defined on $\mathcal{M}$.

With a Hodge star operator and an integration over $\mathcal{M}$, we define the scalar product on $\Omega^\bullet(\mathcal{M})$. Let $\omega \in \Omega^q(\mathcal{M})$ and $\eta \in \Omega^r(\mathcal{M})$, then:

$$\langle\omega, \eta\rangle = \int_{\mathcal{M}} \omega \wedge \star\eta \in \mathbb{R} \quad (1.1.17)$$

With respect to this scalar product, differential forms of distinct degrees, *i.e.* for $q \neq r$, are orthogonal.

1.2 Principal fiber bundles

The theory of principal bundles is an essential feature to construct YM models of gauge field theories. These are related to both the theory of connections on principal bundles and the action of the gauge group. On transitive Lie algebroids, these two points are adapted to a more general scheme. Constructions presented in this section will be refereed, in the next chapters, as the "usual" ones.

1.2.1 Definition

A *principal fiber bundle* $\mathcal{P}(\mathcal{M}, G)$, or *principal bundle*, is a fiber bundle $\mathcal{P} \to \mathcal{M}$, equipped with a *structure group* G, which is a Lie group, so that G acts freely on $\mathcal{P}$ on the right as $\mathcal{P} \times G \ni (u, g) \mapsto R_g u = u{\cdot}g$.[5] The structure group acts "vertically", or "along the fiber", so that, with respect to the map $\pi : \mathcal{P} \to \mathcal{M}$, one has $\pi \circ R_g = \pi$ for any $g \in G$.

[5] Then, $u{\cdot}g = u{\cdot}g'$ if and only if $g = g'$.

The base manifold $\mathcal{M}$ can be seen as the quotient of the principal bundle $\mathcal{P}$ with the structure group G *i.e.* $\mathcal{M} = \mathcal{P}/G$. From our point of view, we consider that the manifold $\mathcal{M}$ "already exists", and $\mathcal{P}$ is then an additional "upper" structure which encodes some inner degrees of freedom. As manifolds, principal bundles can be described with respect to local charts. Such descriptions are not considered in this PhD thesis, we reserve these local charts only for the base manifold $\mathcal{M}$.

1.2.2 Local trivializations of $\mathcal{P}$

Local sections of $\mathcal{P}$ permit to locally describe objects, globally defined on $\mathcal{P}$, in terms of *fields* on $\mathcal{M}$, *i.e.* functions defined on an open set $\mathcal{U} \subset \mathcal{M}$ with values in an arbitrary space. These local descriptions are given by the *local trivializations* of $\mathcal{P}$, and changes of trivializations are related to what will be called the *passive* gauge transformations associated to $\mathcal{P}$.

The local trivialization of a principal bundle over an open set $\mathcal{U}_i$ is given by an invertible diffeomorphism $\phi_i : \mathcal{U}_i \times G \to \pi^{-1}(\mathcal{U}_i)$. It is easier to describe local trivialization in terms of local maps $\phi_{i,p} : G \to \mathcal{P}$ for any $p \in \mathcal{U}_i$, as $\phi_{i,p}(g) = \phi_i(p, g)$. Over the open set $\mathcal{U}_{ij}$, we denote by $g_i = \phi_{i,p}^{-1}(u)$ and $g_j = \phi_{j,p}^{-1}(u)$, with $p = \pi(u)$. *Transition functions* $g_{ij} : \mathcal{U}_{ij} \to G$ are defined as $g_{ij}(p) = g_j{\cdot}g_i^{-1}$ for any $p \in \mathcal{U}_{ij}$. The transitions functions g_{ij} fulfill the structure conditions

$$\begin{cases} g_{ii} = e \\ g_{ji} = g_{ij}^{-1} \\ \text{On } \mathcal{U}_{ijk}, \text{ one has } g_{kj} \circ g_{ji} = g_{ki} \end{cases} \tag{1.2.1}$$

These relations are similar to (1.1.1). Local cross-sections, or sections, of $\mathcal{P}$ are given by the maps $s_i : \mathcal{U}_i \to \mathcal{P}$ defined as $s_i(p) = \phi_{i,p}(e)$. This description, in terms of maps locally defined on the base manifold with values in a fiber, is close to the formalism of gauge field theories. Any element $u \in \pi^{-1}(\mathcal{U}_i)$ can be written as $u = s_i(p)g_i(p)$, where $p = \pi(u)$. Over $\mathcal{U}_{ij}$, one uses the transitions functions $g_{ij} : \mathcal{U}_{ij} \to G$ to obtain the *gluing relation*

$$s_i = s_j{\cdot}g_{ij} \tag{1.2.2}$$

The set of every local cross-sections $(s_i)_{i\in I}$ forms a *system of local cross-sections* of $\mathcal{P}$.

1.2.3 Associated vector bundle

In gauge field theories, scalar fields defined on $\mathcal{M}$ with values in a vector space are geometrically interpreted as sections of an associated fiber bundle.

Let $\mathcal{F}$ be a vector space equipped with a representation of G, *i.e.* a map $\ell : G \to \mathrm{End}(\mathcal{F})$ which preserves the multiplication *i.e.* $\ell_{g_1 g_2} = \ell_{g_1} \circ \ell_{g_2}$, for any $g_1, g_2 \in G$. The *associated vector bundle* $\mathcal{E}^{\mathcal{P}}$ is defined as the quotient fiber bundle $(\mathcal{P} \times \mathcal{F})/G$ where $\mathcal{F}$ is a vector space for the right action ℓ of G. This quotient is defined by identifying elements $(u, f) \in \mathcal{P} \times \mathcal{F}$ with elements $(u, f){\cdot}g = (u{\cdot}g, \ell_{g^{-1}} f)$ for any $g \in G$. This associated fiber bundle can be also denoted by $\mathcal{E}^{\mathcal{P}} = \mathcal{P} \times_\ell \mathcal{F}$. Written under this form, associated vector bundles are given by pairs of elements (u, f) identified by the relation $(u{\cdot}g, f) = (u, \ell_g f)$ for any $g \in G$. The equivalents pairs $(u, f) \in \mathcal{P} \times \mathcal{F}$ are written under the form $[(u, f)]$.

Let s be a G-equivariant map $\mathcal{P} \to \mathcal{F}$ *i.e.* such that $s : \mathcal{P} \to \mathcal{F}$ with $s(u{\cdot}g) = \ell_{g^{-1}} s(u)$ for any $g \in G$. This map denotes a global section of the associated vector bundle. Such a

G-equivariant map defines a global section σ of the associated fiber bundle $\mathcal{E}^{\mathcal{P}}$ as $\sigma(p) = [(u, s(u))]$ for any $u \in \mathcal{P}$. It is obvious that the quantity $[(u, s(u))]$ does not depend on $u = \pi^{-1}(p)$.

The G-equivariant map $s : \mathcal{P} \to \mathcal{F}$ can be locally trivialized over $\mathcal{U}_i$ to a local map $\widetilde{s}_i : \mathcal{U}_i \to \mathcal{F}$ by using the pull-back by a local section σ_i of $\mathcal{P}$. This map can be written as $\widetilde{s}_i(p) = s(\sigma_i(p))$ for any $p \in \mathcal{U}_i$. This map corresponds exactly to the gauge scalar fields in gauge fields theories.

1.2.4 Vector fields on $\mathcal{P}$

We define on a principal bundle $\mathcal{P}(\mathcal{M}, G)$ the space of vector fields $\Gamma(T\mathcal{P})$ as the assignment of a vector $X_u \in T_u\mathcal{P}$ to any point $u \in \mathcal{P}$. Vector fields are defined, at $u \in \mathcal{P}$, in terms of flows as $X_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Phi_{X,t}(u)$, where $\Phi_{X,t}(u) \in \mathcal{P}$ with $t \in [-1, 1]$, and $\Phi_{X,0}(u) = u$ is the flow associated to the vector $X_u \in T_u\mathcal{P}$.

We denote by $C^\infty(\mathcal{P})$ the set of functions $f : \mathcal{P} \to \mathbb{R}$ defined in the neighborhood of $u \in \mathcal{P}$. Vector fields are represented on $C^\infty(\mathcal{P})$ through the pull-back action by the flow *i.e.*

$$(X{\cdot}f)(u) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} f(\Phi_{X,t}(u)) \tag{1.2.3}$$

for any $X \in \Gamma(T\mathcal{P})$ and $f \in C^\infty(\mathcal{P})$. The Lie bracket $[\cdot,\cdot] : \Gamma(T\mathcal{P}) \times \Gamma(T\mathcal{P}) \to \Gamma(T\mathcal{P})$ is defined on $\Gamma(T\mathcal{P})$ by the commutators of two vector fields. In terms of flows, this Lie bracket is defined as

$$([X, Y])_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \left(\frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \Phi_{X,-t} \circ \Phi_{Y,s} \circ \Phi_{X,t}(u) \right) \tag{1.2.4}$$

for any $X, Y \in \Gamma(T\mathcal{P})$ and $u \in \mathcal{P}$.

On principal bundles, *linear tangent applications* are defined as applications between tangent spaces of $\mathcal{P}$. Here, we take the point of view of the sections. Then, linear tangent applications are substituted by $C^\infty(\mathcal{P})$-linear homeomorphisms on the space of vector fields of $\mathcal{P}$.

The structure group acts by right-action on vector fields $\Gamma(T\mathcal{P})$ as $T_*R_g : \Gamma(T\mathcal{P}) \to \Gamma(T\mathcal{P})$. This right-action of G on any $X \in \Gamma(T\mathcal{P})$ is defined as

$$(T_*R_g{\cdot}X)_{u\cdot g} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \Phi_{X,t}(u){\cdot}g \tag{1.2.5}$$

for any $u \in \mathcal{P}$ and $g \in G$, where $\Phi_{X,t}(u)$ is the flow associated to X. A vector field $X \in \Gamma(T\mathcal{P})$ is said to be a *right-invariant vector field* if, for any $u \in \mathcal{P}$, the flow $\Phi_{X,t}(u)$ associated to the vector X_u is such that $\Phi_{X,t}(u{\cdot}g) = \Phi_{X,t}(u){\cdot}g$, for any $g \in G$. An equivalent formulation consists in defining the right-invariant vector fields $X \in \Gamma(T\mathcal{P})$ as the set of vector fields X such that $(T_*R_g{\cdot}X)_{u\cdot g} = X_{u\cdot g}$, for any $u \in \mathcal{P}$ and $g \in G$. Right-invariant vector fields form a vector space, and a $C^\infty(\mathcal{M})$-module, which is denoted by $\Gamma_G(\mathcal{P})$.

The map $\pi : \mathcal{P} \to \mathcal{M}$ acts on vector fields of $\mathcal{P}$ by using the linear application $T_*\pi : \Gamma(T\mathcal{P}) \to \Gamma(T\mathcal{M})$, defined as :

$$(T_*\pi{\cdot}X)_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \pi(\Phi_{X,t}(u)) \tag{1.2.6}$$

such that $p = \pi(u)$ with $p \in \mathcal{M}$ and $u \in \mathcal{P}$. With respect to the composition law $T_*\pi \circ T_*R_g = T_*(\pi \circ R_g) = T_*\pi$, only right-invariant vector fields $X \in \Gamma_G(\mathcal{P})$ can be projected to $\Gamma(T\mathcal{M})$. This result will be used to define the Atiyah Lie algebroids in section 2.1.4.

The space of *vertical vector fields* is defined as the kernel of the map $T^*\pi$. We denote it by $\Gamma(V\mathcal{P})$. Their associated flows $\Phi_{X,t}$ fulfill the conditions $\Phi_{X,t}(u) = u{\cdot}g_t$ where $g_t \in G$ is a connected path in G parametrized by $t \in [-1, 1]$. Vertical vector fields form a vector space, and is also a Lie algebra for the commutator of vector fields and

Assume that G is a connected Lie group with Lie algebra $\mathfrak{g}$. By using the exponential map $\exp : \mathfrak{g} \to G$, any element $g \in G$ can be written as $g = \exp(t\lambda)$ where $t \in \mathbb{R}$ and $\lambda \in \mathfrak{g}$. [6] Then, vertical vector fields X_u can be written in terms of flows as

$$X_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} u{\cdot} \exp(t\lambda) \tag{1.2.7}$$

Then, the *fundamental vector field* $\lambda^\# \in \Gamma(V\mathcal{P})$, associated to $\lambda \in \mathfrak{g}$, is the unique vertical vector field defined in terms of flows as

$$\lambda^\#{}_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} u{\cdot} \exp(t\lambda) \tag{1.2.8}$$

for any $u \in \mathcal{P}$.

Locally, over the open set $\mathcal{U}_i$, we defined the push-forward by a local cross-section $s_i : \mathcal{U}_i \to \mathcal{P}$ of vectors defined on $\mathcal{U}_i$ to vectors defined on $\mathcal{P}$. To any $X \in T_p\mathcal{U}_i$, the local tangent application $s_{i\,*} : T_p\mathcal{U}_i \to T_{s_i(p)}\mathcal{P}$ is defined in terms of flows as

$$(s_{i\,*}X)_{s_i(p)} = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} s_i(\Phi_{X,t}(p))$$

where $\Phi_{X,t}(p)$ is the flow associated to the vector X_p.

1.2.5 Differential forms on $\mathcal{P}$

Differential q-forms are $C^\infty(\mathcal{P})$-multilinear antisymmetric maps $\wedge^*\Gamma(T\mathcal{P}) \to C^\infty(\mathcal{P})$. We denote the associated graded differential complex by $(\Omega^\bullet(\mathcal{P}), \mathrm{d}_\mathcal{P})$, where $\mathrm{d}_\mathcal{P}$ is the Koszul derivative which increases by 1 the degree of forms in $\Omega^r(\mathcal{P})$, associated to the representation of $\Gamma(T\mathcal{P})$ on $C^\infty(\mathcal{P})$.

Let $\mathfrak{g}$ be the Lie algebra of G. We denote by $\Omega^q(\mathcal{P}, \mathfrak{g})$ the set of $C^\infty(\mathcal{P})$-multilinear antisymmetric maps $\wedge^*\Gamma(T\mathcal{P}) \to C^\infty(\mathcal{P}) \otimes \mathfrak{g}$. The differential complex $\Omega^\bullet(\mathcal{P}, \mathfrak{g})$ can be written as $\Omega^\bullet(\mathcal{P}) \otimes \mathfrak{g}$. Then, the Koszul differential is well-defined on $\Omega^\bullet(\mathcal{P}, \mathfrak{g})$, it increases by 1 the degree of forms $\Omega^r(\mathcal{P}, \mathfrak{g})$, where the representation of $\Gamma(T\mathcal{P})$ is only defined on the "$\Omega^\bullet(\mathcal{P})$-part" of $\Omega^\bullet(\mathcal{P}, \mathfrak{g})$.

The induced right-action of an element $g \in G$ on a differential q-form $\omega \in \Omega^q(\mathcal{P})$ is defined by the map $T^*R_g : \Omega^q(\mathcal{P}) \to \Omega^q(\mathcal{P})$ as

$$(T^*R_g\omega)(X_1, \dots, X_q) := \omega(T_*R_g{\cdot}X, \dots, T_*R_g{\cdot}X) \tag{1.2.9}$$

[6] If we consider matricial group, the exponential map can be conveniently written as

$$\exp(t\lambda) = \sum_{m=0} \frac{1}{m!}(t\lambda)^m \quad \forall\lambda \in \mathfrak{g},\ t \in \mathbb{R}$$

for any $X_1, \dots, X_q \in \Gamma(T\mathcal{P})$. Let $\rho : G \to \mathrm{Aut}(\mathfrak{g})$ be a representation, possibly trivial, of G on its Lie algebra $\mathfrak{g}$. The induced right-action of G on elements of $\Omega^q(\mathcal{P}, \mathfrak{g})$ is defined as

$$(T^* R_g \omega)(X_1, \dots, X_q) := \rho(g) \cdot \omega(T_* R_g \cdot X, \dots, T_* R_g \cdot X) \tag{1.2.10}$$

for any $X_1, \dots, X_q \in \Gamma(T\mathcal{P})$.

1.3 Theory of connections on $\mathcal{P}$

We give the definition of connections on $\mathcal{P}$ in terms of elements of $\Omega^1(\mathcal{P}, \mathfrak{g})$. Covariant derivatives associated to connection 1-forms on $\mathcal{P}$ are defined in terms of 1-forms defined on $\mathcal{P}$ with values in the space of first order derivatives of sections of a fiber bundle $\mathcal{E}$. Also, the curvature of a connection on $\mathcal{P}$ is defined as an element of $\Omega^2(\mathcal{P}, \mathfrak{g})$. Both the passive and the active actions of the gauge group are also detailed.

1.3.1 Connections on principal fiber bundles

The subspace of vertical vector fields is defined as the kernel of $T_*\pi$. However, its complementary space in $\Gamma(T\mathcal{P})$, the so-called *horizontal subspace*, cannot be uniquely defined: it is related to a choice of a connection on $\mathcal{P}$. From the geometric point of view, a *connection* on $\mathcal{P}$ is the assignment of a subspace $H\mathcal{P}$ of the tangent bundle $T\mathcal{P}$ such that, for any $u \in \mathcal{P}$, we have:

- $T_u\mathcal{P} = V_u\mathcal{P} \oplus H_u\mathcal{P}$.
- $H_u\mathcal{P}$ is a right invariant vector *i.e.* $T_u R_g : H_u\mathcal{P} \xrightarrow{\simeq} H_{u\cdot g}\mathcal{P}$, for any $u \in \mathcal{P}$.

Then, given a connection, any vector field $X \in \Gamma(T\mathcal{P})$ can be decomposed as $X = X^H + X^V$, where $X^H \in \Gamma(H\mathcal{P})$ and $X^V \in \Gamma(V\mathcal{P})$ are the horizontal and the vertical components of X, respectively.

From a more algebraic point of view, connections on $\mathcal{P}$ are equivalently defined, by differential 1-forms ω defined on $\Gamma(T\mathcal{P})$ with values in $\mathfrak{g}$. Those are called the *Ehresmann connection 1-forms*, or *connection 1-forms*, on $\mathcal{P}$. A connection 1-form ω satisfies the following conditions:

- Let $\lambda^\#$ be the fundamental vector field associated to $\lambda \in \mathfrak{g}$. Then, for any λ, we have $\omega(\lambda^\#) = \lambda$.
- Let $g \in G$, the *induced right action of* G on ω is given by:

$$T^* R_g \cdot \omega = \mathrm{Ad}_{g^{-1}} \circ \omega \tag{1.3.1}$$

Conversely, any element of $\Omega^1(\mathcal{P}, \mathfrak{g})$ which fulfills these two conditions defines a connection on $\mathcal{P}$. The space of horizontal vector fields $\Gamma(H\mathcal{P})$ is defined as the kernel of the connection 1-form ω *i.e.* $\Gamma(H\mathcal{P}) = \{X \in \Gamma(T\mathcal{P}) | \, \omega(X) = 0\}$.

1.3.2 Covariant derivative and curvatures

Let $h : \Gamma(T\mathcal{P}) \to \Gamma(H\mathcal{P})$ be a projector on the horizontal subspace associated to a connection on $\mathcal{P}$. Let α be a differential q-form on $\mathcal{P}$ with values in sections of a fiber bundle $\mathcal{E}$. The *covariant derivative* associated to this connection is a map $\mathcal{D}$ acting on q-forms α as

$$(\mathcal{D}\alpha)(X_1, \dots, X_q) = (\mathrm{d}\alpha)(h \circ X_1, \dots, h \circ X_q) \tag{1.3.2}$$

for any $X_1, \dots, X_q \in \Gamma(T\mathcal{P})$. Obviously, the covariant derivative vanishes for vertical vector fields, due to the presence of the projector h. This definition will be used in chapter 4 to define the covariant derivative in the context of transitive Lie algebroids.

The *curvature* R associated to a connection 1-form is an element of $\Omega^2(\mathcal{P}, \mathfrak{g})$ which can be defined by two equivalent distinct ways. The former consists in defining R as the covariant derivative of the connection 1-form *i.e.* as $R(X,Y) = (\mathrm{d}\omega)(h \circ X, h \circ Y)$, for any $X, Y \in \Gamma(T\mathcal{P})$. The latter is given in terms of the connection 1-form ω, by the Cartan equation structure, as

$$R(X,Y) = \mathrm{d}\omega(X,Y) + [\omega(X), \omega(Y)] \tag{1.3.3}$$

for any $X, Y \in \Gamma(T\mathcal{P})$. Here the Lie bracket is a graded Lie bracket defined on the space of differential forms $\Omega^\bullet(\mathcal{P}, \mathfrak{g})$. [7] By construction, R is *horizontal*, *i.e.* $R(X,Y) = 0$ if any of the vector fields X or Y are vertical vector fields.

1.3.3 Local expression of a connection 1-form

Connection 1-forms are locally trivialized as elements of $\Omega^1(\mathcal{U}_i, \mathfrak{g})$ by using local cross-sections $(s_i)_{i\in I}$ on $\mathcal{P}$. Then, the local trivialization of ω over $\mathcal{U}_i$ is given by the $C^\infty(\mathcal{U}_i)$-linear map $\omega_{\mathrm{loc},\, i} : \Gamma(T\mathcal{U}_i) \to \mathfrak{g}$ as

$$\omega_{\mathrm{loc},\, i}(X) = (s_i^* \omega)(X) = \omega(s_{i\,*} X) \tag{1.3.4}$$

for any $X \in \Gamma(T\mathcal{U}_i)$. Over the open set $\mathcal{U}_{ij}$, the gluing transformations of the connection 1-form are

$$\omega_{\mathrm{loc},\, i} = g_{ij}^{-1} \omega_{\mathrm{loc},\, i} g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij} \tag{1.3.5}$$

where d is the Koszul derivative and $g_{ij}^{-1}\mathrm{d}g_{ij}$ is a $\mathfrak{g}$-valued 1-form defined on $\mathcal{U}_{ij}$. We define the local trivialization of the curvature of ω over $\mathcal{U}_i$ as

$$R_{\mathrm{loc},\, i} = \mathrm{d}\omega_{\mathrm{loc},\, i} + \frac{1}{2}[\omega_{\mathrm{loc},\, i}, \omega_{\mathrm{loc},\, i}] \tag{1.3.6}$$

where the differential operator d is the Koszul derivative acting on $\Omega^\bullet(\mathcal{M})$. The local expressions $R_{\mathrm{loc},\, i}$ and $R_{\mathrm{loc},\, j}$ of the curvature R, associated to ω, are related over $\mathcal{U}_{ij}$ by the homogeneous passive gauge transformation

$$R_{\mathrm{loc},\, i} = g_{ij}^{-1} R_{\mathrm{loc},\, j} g_{ij} \tag{1.3.7}$$

These gauge transformations are induced by changes of local trivialization of $\mathcal{P}$. They form the set of the *passive gauge transformation*.

[7] The graded Lie bracket on $\Omega^\bullet(\mathcal{P}, \mathfrak{g})$ is defined as $[\omega_1 \otimes X_1, \omega_2 \otimes X_2] = \omega_1 \wedge \omega_2 \otimes [X_1, X_2]$ for any $\omega_1, \omega_2 \in \Omega^\bullet(\mathcal{P})$ and $X_1, X_2 \in \mathfrak{g}$.

1.3.4 Active gauge transformations on $\mathcal{P}$

The gauge group $\mathcal{G}$ associated to a principal bundle $\mathcal{P}(\mathcal{M}, G)$ is defined as the group of vertical automorphisms $f : \mathcal{P} \to \mathcal{P}$, along the fiber $\mathcal{P}$, compatible with the action of G in the sense that $f(u{\cdot}a) = f(u){\cdot}a$ for any $a \in G$ and $(\pi \circ f) = \pi$. This automorphism can be written in terms of maps $g : \mathcal{P} \to G$ as $f(u) = u{\cdot}g(u)$, for any $u \in \mathcal{P}$. With respect to the right-action of G, the map $g : \mathcal{P} \to G$ transforms as $g(u{\cdot}a) = a^{-1}g(u)a$, for any $a \in G$. This transformation indicates that g defines a section of the associated fiber bundle $\mathcal{P} \times_\beta G$, where β acts by conjugacy on G as $\beta_g g' = g^{-1}g'g$ for any $g, g' \in G$. We denote by $\mathcal{G}$ the section of the associated fiber bundle $\mathcal{P} \times_\beta G$. Depending on the context, elements of the gauge group are either G-equivariant maps $\mathcal{P} \to G$, or, given a local section of $\mathcal{P}$, a field $\mathcal{U} \to G$ (see subsection 1.2.3 for the correspondence).

The gauge group $\mathcal{G}$ acts on connection 1-forms ω as

$$\omega \mapsto \omega^g = g^{-1}\omega g + g^{-1}\mathrm{d}g \tag{1.3.8}$$

where $g \in \mathcal{G}$. This transformation corresponds to an *active gauge transformation* of $\mathcal{G}$. It its straightforward to check that ω^g still defines a connection 1-form on $\mathcal{P}$. In this sense, we say that the space of connection 1-forms is compatible with the action of the gauge group. Gauge transformations of ω induce gauge transformations of the curvature R. It transforms as $R^g = g^{-1}Rg$ for any $g \in G$.

In the theory of fiber bundles, active and passive gauge transformations have the same mathematical expression. We will see that it is no longer true in section 6.3: on transitive Lie algebroids, we define an “algebraic” infinitesimal action of the gauge group, distinct from the “geometric” one. This is motivated by the introduction of generalized connections.

Chapter 2

The theory of Lie algebroids

The theory of Lie algebroids was initially constructed as the infinitesimal version of the Lie groupoids. Lie algebroids were introduced by Pradines [Pra67] and I refer the reader to [Mac05] for the full development of the theory (a pedagogical construction, from groupoids to algebroids, is presented in [Wei96]). Here, I give an algebraic definition of Lie algebroids so that it will be easier to construct algebraic structures on it. It also permits to establish conceptual links with the differential structures of the "usual" geometry already exhibited in the chapter 1. Indeed, through this chapter, we consider the framework of the Lie algebroids as an extension of the geometry of manifolds. This extension is given by the consideration of algebraic degrees of freedom in addition to the geometric degrees of freedom defined by the geometry of the vector fields. Nevertheless, in the general case, this extension is not canonically identified and a system of local trivializations of Lie algebroids is required to give a concrete description of these two spaces. For the reader who is already familiar with the Lie algebroids, this chapter is the opportunity for me to stress the basic facts needed for the upcoming constructions of the next chapters. Commentaries and discussions will highlight the relevance of these structures.

2.1 Basic notions

2.1.1 Definitions

One gives the definition of a Lie algebroid defined over a manifold $\mathcal{M}$, used in [Mac05].

A *Lie algebroid* $\mathcal{A}$ defined over $\mathcal{M}$ is a vector bundle $\mathcal{A} \xrightarrow{\pi} \mathcal{M}$ equipped with a vector bundle map $\rho : \mathcal{A} \to T\mathcal{M}$ called the *anchor* of $\mathcal{A}$, and a Lie bracket $[\cdot,\cdot]$ defined on the space of the section $\Gamma(\mathcal{A})$ which is $\mathbb{R}$-linear, antisymmetric and respects the Jacoby identity. This Lie bracket $[\cdot,\cdot] : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ fulfills the relation

$$[\mathfrak{X}, f{\cdot}\mathfrak{Y}] = f{\cdot}[\mathfrak{X},\mathfrak{Y}] + (\rho(\mathfrak{X}){\cdot}f){\cdot}\mathfrak{Y} \quad \text{and} \quad \rho([\mathfrak{X},\mathfrak{Y}]) = [\rho(\mathfrak{X}), \rho(\mathfrak{Y})] \tag{2.1.1}$$

for any $\mathfrak{X},\mathfrak{Y} \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$. The anchor ρ is a morphism of Lie algebras. Usually, one uses distinguished symbols to denote the Lie bracket over $\Gamma(T\mathcal{M})$ and the Lie bracket on $\Gamma(\mathcal{A})$. Here, I use the same symbol $[\cdot,\cdot]$ for both of them since the context is sufficiently clear.

One denotes by $\mathsf{A} = \Gamma(\mathcal{A})$ the space of sections of the vector bundle $\mathcal{A}$ *i.e.* the space of maps $\mathfrak{X} : \mathcal{M} \to \mathcal{A}$. The space A is a module over the space of functions $C^\infty(\mathcal{M})$ and the anchor ρ is naturally extended to a map of $C^\infty(\mathcal{M})$-modules $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$. In the rest of this PhD thesis, one considers directly the space of sections of $\mathcal{A}$, so that one abusively calls $\mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M})$ a Lie algebroid over $\mathcal{M}$.

This definition shows that Lie algebroids are connected to the differential geometry of $\mathcal{M}$ in the sense that the map ρ identifies elements of A with vector fields on $\mathcal{M}$. However, given a Lie algebroid, it is not possible to canonically identify the elements which are related to these vector fields. This remark is similar to the statement that, in the theory of the fiber bundles, we cannot globally split the *inner degrees of freedom* from the *geometric degrees of freedom*. To do so, one needs a system of local trivializations of the fiber bundle. Local trivialization of Lie algebroids will be detailed in section 2.2. A Lie algebroid can be restricted over any open set $\mathcal{U} \subset \mathcal{M}$ and this restriction is denoted by $\mathsf{A}|_{\mathcal{U}} = \Gamma(\mathcal{A}|_{\mathcal{U}})$ where $\mathcal{A}|_{\mathcal{U}}$ is the restriction of the vector bundle $\mathcal{A}$ over $\mathcal{U}$. Accordingly to this restriction, the map ρ is restricted to $\rho_{\mathcal{U}} : \mathsf{A}|_{\mathcal{U}} \to \Gamma(T\mathcal{U})$ so that the Lie bracket preserves $\mathsf{A}|_{\mathcal{U}}$.

Depending on the nature of the map $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$ (surjective, null or of locally constant rank), one defines different kinds of Lie algebroids. We are concerned with the case where ρ is surjective, then we can think of A_p as a "fiber space" over the tangent space $T_p\mathcal{M}$, for any $p \in \mathcal{M}$.

A *transitive Lie algebroid* is a Lie algebroid $\mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M})$ whose anchor ρ is surjective. The kernel of the map ρ is denoted by L and is called the *kernel of the transitive Lie algebroid*. The Lie bracket on L inherits from the Lie bracket on A and one has:

$$[\cdot,\cdot] : \mathsf{L} \times \mathsf{L} \to \mathsf{L} \quad ; \quad [\ell, f{\cdot}\ell'] = f{\cdot}[\ell, \ell'] \tag{2.1.2}$$

for any $f \in C^\infty(\mathcal{M})$ and $\ell, \ell' \in \mathsf{L}$. Here, I use the same bracket notations for both A and L.

Actually, the kernel L can be defined as the space of sections of the vector bundle $\mathcal{L}$, which is the kernel of $\rho : \mathcal{A} \to T\mathcal{M}$. The vector bundle $\mathcal{L}$ will be concerned with the section 5.2.5 so that the reader should keep in mind the existence of this underlying space.

It is more convenient to see L as a *totally intransitive* Lie algebroid defined over $\mathcal{M}$ *i.e.* with anchor $\rho = 0$. The Lie algebroid L is injected in the transitive Lie algebroid A by the injective morphism of Lie algebras $\iota : \mathsf{L} \to \mathsf{A}$. We summarize the definition of a transitive Lie algebroid by the following short exact sequence of $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow \mathsf{L} \xrightarrow{\ \iota\ } \mathsf{A} \xrightarrow{\ \rho\ } \Gamma(T\mathcal{M}) \longrightarrow 0 \tag{2.1.3}$$

Elements in the kernel of ρ will be considered either as elements of A or as elements of L, so that their descriptions are given *modulo* the map ι.

With respect to the Lie bracket, Lie algebroids A and L obeys the following relations

$$[\mathsf{A}, \mathsf{A}] \subset \mathsf{A} \quad ; \quad [\mathsf{A}, \mathsf{L}] \subset \mathsf{L} \quad ; \quad [\mathsf{L}, \mathsf{L}] \subset \mathsf{L} \tag{2.1.4}$$

By convention, for any $\ell \in \mathsf{L}$ and $\mathfrak{X} \in \mathsf{A}$, the element $[\ell, \mathfrak{X}]$ is the unique element in L such that $\iota([\ell, \mathfrak{X}]) = [\iota(\ell), \mathfrak{X}]$. The kernel L is an *ideal* in A.

Contrary to elements of A related to the space of vector fields $\Gamma(T\mathcal{M})$ by the anchor map, the space L is explicitly identified as the kernel of ρ, so that it can be canonically exhibited as a sub Lie algebra of A. This remark is analogous to the case of vertical vector fields on a principal bundle $\mathcal{P} \xrightarrow{\pi} \mathcal{M}$. These are defined as the kernel of the linear tangent operator $T_*\pi : \Gamma(T\mathcal{P}) \to \Gamma(T\mathcal{M})$ and then they form a well-defined subspace of $\Gamma(T\mathcal{P})$. This is not the case for its complementary space in $\Gamma(T\mathcal{P})$, the horizontal subspace, which cannot be canonically identified without use of a connection.

In the context of transitive Lie algebroids, this duality between "vertical" and "horizontal" spaces is directly encoded in the definition of the maps ι and ρ. Indeed, the map ι is an injective map *from* L *to* A whereas ρ is a surjective map *from* A *to* $\Gamma(T\mathcal{M})$, as depicted in the sequence (2.1.3). In section 4.1.1, we will see that a connection defined on A is merely defined as a splitting of this short exact sequence *i.e.* a map going *from* $\Gamma(T\mathcal{M})$ *to* A such that $\rho \circ \nabla = \mathrm{Id}_{\Gamma(T\mathcal{M})}$.

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L} \arrow[r, "\iota"] & \mathsf{A} \arrow[r, "\rho"] & \Gamma(T\mathcal{M}) \arrow[r] \arrow[l, bend left, "\nabla"] & 0 \end{tikzcd} \tag{2.1.5}$$

2.1.2 Morphism of Lie algebroids

This section gives the definition of the isomorphism between two transitive Lie algebroids A and B. Such isomorphisms will be used for the local description of transitive Lie algebroids in section 2.2.

Let $\mathsf{A} \xrightarrow{\rho_\mathsf{A}} \Gamma(T\mathcal{M})$ and $\mathsf{B} \xrightarrow{\rho_\mathsf{B}} \Gamma(T\mathcal{M})$ two transitive Lie algebroids with kernels L_A and L_B and with injective maps ι_A and ι_B, respectively. A *morphism of Lie algebroids* $\varphi : \mathsf{A} \to \mathsf{B}$ is a $C^\infty(\mathcal{M})$-linear morphism of Lie algebras which is *base-preserving* and compatible with the anchors *i.e.*

$$\varphi([\mathfrak{X}, \mathfrak{Y}]) = [\varphi(\mathfrak{X}), \varphi(\mathfrak{Y})] \quad ; \quad \rho_\mathsf{B} \circ \varphi = \rho_\mathsf{A} \tag{2.1.6}$$

for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$. A morphism of Lie algebroids $\varphi_\mathsf{L} : \mathsf{L}_\mathsf{A} \to \mathsf{L}_\mathsf{B}$ is induced on the respective kernels by the relation $\iota_\mathsf{B} \circ \varphi_\mathsf{L}(\ell_\mathsf{A}) = \varphi \circ \iota_\mathsf{A}(\ell_\mathsf{A})$, with $\ell_\mathsf{A} \in \mathsf{L}_\mathsf{A}$. Then, one has the following *commutative diagram*:

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L}_\mathsf{A} \arrow[r, "\iota_\mathsf{A}"] \arrow[d, "\varphi_\mathsf{L}"'] & \mathsf{A} \arrow[r, "\rho_\mathsf{A}"] \arrow[d, "\varphi"] & \Gamma(T\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathsf{L}_\mathsf{B} \arrow[r, "\iota_\mathsf{B}"] & \mathsf{B} \arrow[r, "\rho_\mathsf{B}"] & \Gamma(T\mathcal{M}) \arrow[r] & 0 \end{tikzcd} \tag{2.1.7}$$

An *isomorphism of Lie algebroids* is a morphism of Lie algebroids $\varphi : \mathsf{A} \to \mathsf{B}$ which is invertible *i.e.* it is a $C^\infty(\mathcal{M})$-linear isomorphism of Lie algebras which is base-preserving and compatible with the anchors.

2.1.3 Trivial Lie algebroids

As a first example of transitive Lie algebroids, one defines the trivial Lie algebroids.

A transitive Lie algebroid is trivial in the sense that both the geometry of vector fields on $\mathcal{M}$ and the algebraic degrees of freedom of the kernel of ρ are described in distinct vector spaces. Then, trivial Lie algebroids are defined as a vectorial sum of $\Gamma(T\mathcal{M})$ and the space of sections of a trivial Lie algebra bundle. The "triviality" of a Lie algebroid is similar with the triviality of a principal bundle when it can be globally defined as the cartesian product between $\mathcal{M}$ and the structure group G. Trivial Lie algebroids are an essential structure that will take part in local trivializations of objects defined on transitive Lie algebroids.

One denotes by $\Gamma(\mathcal{M} \times \mathfrak{g})$ the space of sections on the trivial Lie algebra bundle $\mathcal{M} \times \mathfrak{g}$ *i.e.* a trivial vector bundle with typical fiber a Lie algebra $\mathfrak{g}$. Elements of $\Gamma(\mathcal{M} \times \mathfrak{g})$ are given by the maps $\gamma : \mathcal{M} \to \mathfrak{g}$. Both the spaces $\Gamma(T\mathcal{M})$ and $\Gamma(\mathcal{M} \times \mathfrak{g})$ are $C^\infty(\mathcal{M})$-modules so that one defines the *trivial Lie algebroid* on $\mathcal{M}$ modeled over $\mathfrak{g}$ by the short exact sequence of $C^\infty(\mathcal{M})$-modules:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathcal{M} \times \mathfrak{g}) \arrow[r, "\iota"] & \Gamma(T\mathcal{M}) \oplus \Gamma(\mathcal{M} \times \mathfrak{g}) \arrow[r, "\rho"] & \Gamma(T\mathcal{M}) \arrow[r] & 0 \end{tikzcd} \tag{2.1.8}$$

with the maps ι and ρ are defined as:

$$\iota : \Gamma(\mathcal{M}\times\mathfrak{g}) \to \Gamma(T\mathcal{M})\oplus\Gamma(\mathcal{M}\times\mathfrak{g}) \quad ; \quad \iota : \gamma \mapsto 0\oplus\gamma \tag{2.1.9}$$
$$\rho : \Gamma(T\mathcal{M})\oplus\Gamma(\mathcal{M}\times\mathfrak{g}) \to \Gamma(T\mathcal{M}) \quad ; \quad \rho : X\oplus\gamma \mapsto X \tag{2.1.10}$$

Trivial Lie algebroid are denoted as $\mathsf{TLA}(\mathcal{M},\mathfrak{g}) = \Gamma(T\mathcal{M})\oplus\Gamma(\mathcal{M}\times\mathfrak{g})$ and the corresponding Lie bracket is defined as

$$[X\oplus\gamma, Y\oplus\eta] = [X,Y]\oplus(X{\cdot}\eta - Y{\cdot}\gamma + [\gamma,\eta]) \tag{2.1.11}$$

for any $X\oplus\gamma, Y\oplus\eta \in \mathsf{TLA}(\mathcal{M},\mathfrak{g})$. The action of vector fields on $\Gamma(\mathcal{M}\times\mathfrak{g})$ is defined as in (1.1.2). Here, one has used the Lie bracket on $\Gamma(T\mathcal{M})$. It is easy to verify that the commutating relations (2.1.4) are fulfilled.

Trivial Lie algebroids give a concrete picture, on the one hand, of the description of the geometric degrees of freedom and, on the other hand, of the description of algebraic degrees of freedom. However, the Lie bracket (2.1.11) defined on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ "mixes" these two kinds of degrees of freedom. This comes from the fact that vector fields on $\mathcal{M}$ are represented on $\Gamma(\mathcal{M}\times\mathfrak{g})$, but the inverse is not true. Thus, the geometric component of a trivial Lie algebroid does not "see" its associated algebraic component. This point will be taken into account in the computation of the gluing relations of global objects.

2.1.4 Atiyah Lie algebroids

A second example of transitive Lie algebroids is the Atiyah Lie algebroid associated to a principal bundle $\mathcal{P}(\mathcal{M},G)$. Here, the principal bundle $\mathcal{P}$ permits to interpret the objects defined on transitive Lie algebroids as objects defined on $\mathcal{P}$. For instance, connections defined on A in section 4.1.1 are exactly the Ehresmann connections on $\mathcal{P}$. Moreover, constructions on transitive Lie algebroids come from constructions on $\mathcal{P}$, such as the infinitesimal action of the gauge group. We will see in the chapter 4.2 that transitive Lie algebroids are equipped with some algebraic structures which cannot be interpreted in terms of the geometry of $\mathcal{P}$.

On Atiyah Lie algebroids, one does not look at all the vector fields on $\mathcal{P}$ but only at those which can be projected to vector fields on $\mathcal{M}$ *i.e.* the space of the right-invariant vector fields on $\mathcal{P}$. In [LM12b], it is shown that right-invariant vector fields form a basis for the vector fields on $\mathcal{P}$ so that any vector field $X\in\Gamma(T\mathcal{P})$ can be uniquely decomposed as $X = f^i\mathfrak{X}_i$ where $f^i\in C^\infty(\mathcal{P})$ and $\mathfrak{X}_i\in\Gamma_G(\mathcal{P})$. Then, Atiyah Lie algebroids are not concerned with the whole space $\Gamma(T\mathcal{P})$ but only with elements of the basis $\Gamma_G(\mathcal{P})$.

Algebraic degrees of freedom of the Atiyah Lie algebroid are encoded in the space $\Gamma_G(\mathcal{P},\mathfrak{g})$ of sections on the associated fiber bundle $\mathcal{P}\times_{\mathrm{Ad}}\mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G and Ad denotes the adjoint action of G on $\mathfrak{g}$ given by the relation $\mathrm{Ad}_g\gamma = g\gamma g^{-1}\in\mathfrak{g}$, for any $g\in G$ and $\gamma\in\mathfrak{g}$. The space of sections on $\mathcal{P}\times_{\mathrm{Ad}}\mathfrak{g}$ is given by the set of maps $\Gamma_G(\mathcal{P},\mathfrak{g}) = \{v:\mathcal{P}\to\mathfrak{g}\,|\,v(u{\cdot}g) = \mathrm{Ad}_{g^{-1}}v(u),\ \forall u\in\mathcal{P} \text{ and } g\in G\}$. This space is identified with the infinitesimal description of the gauge group.

The vector spaces $\Gamma_G(\mathcal{P})$ and $\Gamma_G(\mathcal{P},\mathfrak{g})$ are $C^\infty(\mathcal{M})$-modules so that one defines the *Atiyah Lie algebroid* by the short exact sequence of $C^\infty(\mathcal{M})$-modules

$$0 \longrightarrow \Gamma_G(\mathcal{P},\mathfrak{g}) \xrightarrow{\iota} \Gamma_G(\mathcal{P}) \xrightarrow{\rho} \Gamma(T\mathcal{M}) \longrightarrow 0 \tag{2.1.12}$$

with the maps $\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(\mathcal{P})$ and $\rho : \Gamma_G(\mathcal{P}) \to \Gamma(T\mathcal{M})$ are defined as:

$$\iota : v \mapsto \left(\iota(v) : u \mapsto \iota(v)_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} u{\cdot}e^{-tv(u)}, \quad \forall u \in \mathcal{P}\right), \quad \forall\, v \in \Gamma_G(\mathcal{P}, \mathfrak{g}) \tag{2.1.13}$$

$$\rho : \mathfrak{X} \mapsto \left(\rho(\mathfrak{X}) : p \mapsto \rho(\mathfrak{X})_p = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \pi(\Phi_{\mathfrak{X},t}(u)), \quad \forall u \in \pi^{-1}(p)\right), \quad \forall\, \mathfrak{X} \in \Gamma_G(\mathcal{P}) \tag{2.1.14}$$

where $\Phi_{\mathfrak{X},t}(u)$ is the flow associated to the right-invariant vector field $\mathfrak{X}$. Up to a minus sign, the injective map $\iota : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \Gamma_G(\mathcal{P})$ gives the fundamental vector field associated to elements of $\Gamma_G(\mathcal{P}, \mathfrak{g})$ *i.e.* $(\iota \circ v)_u = (-v(u))^{\#}|_u$, for any $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$ and $u \in \mathcal{P}$.

With geometric computations, it is straightforward to verify that the commutating relations (2.1.4) are fulfilled. In particular, one computes the Lie bracket of a right-invariant vector field $\mathfrak{X} \in \Gamma_G(\mathcal{P})$ with a vertical vector field $\iota \circ v \in \Gamma_G(\mathcal{P})$. For any $u \in \mathcal{P}$, one has:

$$[\mathfrak{X}, \iota \circ v]_u = (\iota \circ (\mathfrak{X}{\cdot}v))_u \tag{2.1.15}$$

This also proves that the space $\iota \circ \Gamma_G(\mathcal{P}, \mathfrak{g})$ is an ideal in $\Gamma_G(\mathcal{P})$, as expected.

2.2 Local trivializations of Lie algebroids

The local trivialization of a Lie algebroid realizes an isomorphism of between a Lie algebroid $\mathsf{A}|_{\mathcal{U}}$, restricted over the open set $\mathcal{U}$, and a trivial Lie algebroid $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ over $\mathcal{U}$ modeled on $\mathfrak{g}$. The geometric and the algebraic degrees of freedom of an element $\mathfrak{X}$ of a Lie algebroid A are then identified as two distinct subspaces and can be written under the form $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Then, transitive Lie algebroids can either be described as a global construction over $\mathcal{M}$, which would correspond to a first level of description, or, locally, as a set of trivial Lie algebroids modeled on $\mathfrak{g}$ which would correspond to a second level of description. This point will be concretely illustrated in the following sections. Providing a local chart of $\mathcal{M}$, we wan define a third level of description of A in terms of elements of $\mathbb{R}^m$.

As a globally-defined object on A, a section $\mathfrak{X}$ is locally described by a collection of elements $(X_i \oplus \gamma_i)_{i \in I}$n which are not equals for any $i \in I$ (otherwise, A would be trivial). Thus, over the open set $\mathcal{U}_{ij}$, the elements $X_i \oplus \gamma_i$ and $X_j \oplus \gamma_j$ can be related by the corresponding gluing relations associated to the system of local trivializations.

Gluing relations of local trivializations of objects defined on a principal bundle $\mathcal{P}$ are well-known (see [KN96a]): they are given by the map $g_{ij} : \mathcal{U}_{ij} \to G$. In the context of the Lie algebroids, such a Lie group G is not defined in the general case so that one computes new gluing relations which are specific to Lie algebroids.

2.2.1 Local trivializations of Lie algebroids

A *local trivialization of Lie algebroid* over an open set $\mathcal{U} \subset \mathcal{M}$ is given by a local isomorphism of Lie algebroids $S : \mathsf{TLA}(\mathcal{U}, \mathfrak{g}) \to \mathsf{A}|_{\mathcal{U}}$ where $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ denotes the trivial Lie algebroid over $\mathcal{U}$ modeled on the Lie algebra $\mathfrak{g}$ and $\mathsf{A}|_{\mathcal{U}}$ denotes the space of sections of the vector bundle $\mathcal{A}$ restricted over $\mathcal{U}$.

With abuse of notation, we use the same symbol ρ to represent the anchor $\rho : \mathsf{A}_{|\mathcal{U}} \to \Gamma(T\mathcal{U})$ and the anchor $\rho : \mathsf{TLA}(\mathcal{U}, \mathfrak{g}) \to \Gamma(T\mathcal{U})$. As an isomorphism of Lie algebroids, the two maps $S : \mathsf{TLA}(\mathcal{U}, \mathfrak{g}) \to \mathsf{A}_{|\mathcal{U}}$ and $S^{-1} : \mathsf{A}_{|\mathcal{U}} \to \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ are morphisms of Lie algebroids which are compatible with the anchors in the sense that $\rho \circ S = \rho$ and $\rho \circ S^{-1} = \rho$.

The map S can be restricted either to the space of vector fields $\Gamma(T\mathcal{U})$ or to the space of sections $\Gamma(\mathcal{U} \times \mathfrak{g})$. At the level of A, the restriction of S^{-1} is only possible on L since its complementary subspace is not canonically defined without connection. At the level of $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the map S splits in two maps $\nabla^0 : \Gamma(T\mathcal{U}) \to \mathsf{A}|_{\mathcal{U}}$ and $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \to \mathsf{L}|_{\mathcal{U}}$. The first map is the restriction of the map S to elements of $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ which are of the form $X \oplus 0$ *i.e.*

$$\nabla^0 : \Gamma(T\mathcal{U}) \to \mathsf{A}_{|\mathcal{U}} \quad ; \quad \nabla^0_X = S(X \oplus 0), \tag{2.2.1}$$

for any $X \in \Gamma(T\mathcal{U})$. This map has the following properties

- $[\nabla^0_X, \nabla^0_Y] = \nabla^0_{[X,Y]}$ for any $X, Y \in \Gamma(T\mathcal{U})$.
- $\nabla^0_{f \cdot X} = f \cdot \nabla^0_X$ for any $f \in C^\infty(\mathcal{U})$ and $X \in \Gamma(T\mathcal{U})$.
- $\rho \circ \nabla^0_X = X$ for any $X \in \Gamma(T\mathcal{U})$.

The second map $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \to \mathsf{L}_{|\mathcal{U}}$ is defined by the relation:

$$\iota \circ \Psi(\gamma) = S(0 \oplus \gamma) \tag{2.2.2}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. This map has the following properties

- $\Psi([\gamma, \eta]) = [\Psi(\gamma), \Psi(\eta)]$ for any $\gamma, \eta \in \Gamma(\mathcal{U} \times \mathfrak{g})$.
- $\Psi(f \cdot \gamma) = f \cdot \Psi(\gamma)$ for any $f \in C^\infty(\mathcal{U})$ and $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$.

Finally, the local trivialization of an element $\mathfrak{X} \in \mathsf{A}$ can be either given by the map S or by the pair of elements (∇^0, Ψ). These two formulations are related by

$$S(X \oplus \gamma) = \nabla^0_X + \iota \circ \Psi(\gamma) \tag{2.2.3}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. One says that an element $\mathfrak{X} \in \mathsf{A}$ is *locally trivialized as* $X \oplus \gamma$ if $\mathfrak{X} = S(X \oplus \gamma)$ and one uses the notation $\mathfrak{X} \underset{\text{loc}}{\simeq} X \oplus \gamma$

The fact that S is an isomorphism of Lie algebroids induces that the inverse map Ψ^{-1} exists and is defined as $\Psi^{-1}(\ell) = S^{-1}(\iota \circ \ell)$, for any $\ell \in \mathsf{L}$, so that the map Ψ is a $C^\infty(\mathcal{U})$-linear isomorphism of Lie algebras. The inverse map Ψ^{-1} exists since the map S^{-1} can be restricted to L but, without a connection on A, it cannot be restricted to its complementary space so that the map $(\nabla^0)^{-1}$ is not mathematically well-defined.

In section 2.1.3, one has noted that the Lie bracket on a trivial Lie algebroid mixes the geometric and the algebraic components of $\mathsf{TLA}(\mathcal{M}, \mathfrak{g})$. In the case of the local trivialization of a transitive Lie algebroid, this observation results in the existence of a *compatibility relation* between the maps ∇^0 and Ψ defined as

$$[\nabla^0_X, \iota \circ \Psi(\gamma)] = \iota \circ \Psi(X \cdot \gamma) \tag{2.2.4}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$.

An *atlas of Lie algebroids* consists in the data of the pairs $(\mathcal{U}_i, S_i)_{i \in I} = (\mathcal{U}_i, \nabla^0_i, \Psi_i)_{i \in I}$. One assumes that the charts $(\mathcal{U}_i)_{i \in I}$ of the atlas of Lie algebroids coincide with the charts of the atlas on $\mathcal{M}$. However, changes of local trivializations of transitive Lie algebroids and changes of local charts are distinct operations *i.e.* these have to be computed independently.

2.2.2 Changes of trivializations

To complete the local description of a transitive Lie algebroid in terms of trivial Lie algebroids, one needs to define the gluing relations associated to the atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I}$. Over the open set $\mathcal{U}_{ij}$, these gluing relations give the relations between the trivializations $X_i \oplus \gamma_i$ and $X_j \oplus \gamma_j$ of the same element $\mathfrak{X} \in \mathsf{A}$. Conversely, the gluing relations permit to identify a family of pairs $(X_i \oplus \gamma_i)_{i\in I}$ as the local trivializations of a global element of A.

Consider $(\mathcal{U}, S)$ and $(\mathcal{U}, S')$ two local trivializations of A over the same open set $\mathcal{U}$. Then, there exists two maps $\alpha \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$ and $\chi \in \Omega^1(\mathcal{U}) \otimes \mathfrak{g}$ such that

$$(S^{-1} \circ S')(X \oplus \gamma) = X \oplus (\alpha(\gamma) + \chi(X)) \tag{2.2.5}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Since S and S' are isomorphisms of Lie algebroids, it is direct to check that $\rho \circ (S^{-1} \circ S') = \rho$, then the tangent vector $X \in \Gamma(T\mathcal{U})$ is invariant by change of local trivializations. By changes of trivializations of A, the "geometric component" of $\mathfrak{X}$ stays the same. However, the algebraic component γ is "moved" by an endomorphism $\alpha : \mathfrak{g} \to \mathfrak{g}$, then"lifted" by the action of the associated vector field X.

On the open set $\mathcal{U}_{ij}$, an element $\mathfrak{X} \in \mathsf{A}$ can be trivialized either as $\mathfrak{X} \underset{\text{loc}}{\simeq} X \oplus \gamma_i \in \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$, with respect to the chart $(\mathcal{U}_i, S_i)$, or as $\mathfrak{X} \underset{\text{loc}}{\simeq} (X \oplus \gamma_j) \in \mathsf{TLA}(\mathcal{U}_j, \mathfrak{g})$, with respect to the chart $(\mathcal{U}_j, S_j)$. The vector field $X \in \Gamma(T\mathcal{U}_{ij})$ is the same for both trivializations of $\mathfrak{X}$ and elements γ_i and γ_j are related by the formula:

$$\gamma_i = \alpha^i_j(\gamma_j) + \chi_{ij}(X) \tag{2.2.6}$$

where $\alpha^i_j : \Gamma(\mathcal{U}_{ij} \times \mathfrak{g}) \to \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$ and $\chi_{ij} : \Gamma(T\mathcal{U}_{ij}) \to \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$. The maps α^i_j and χ_{ij} are called the *gluing functions of the Lie algebroid*. By convention, one designates with an upper index the image space of α and by a lower index its source space. Concerning the map χ, one puts the index of the space of arrival to the left of the index of the source space. With these notations, it is straightforward to check the compatibility relations

$$\alpha^i_i = \mathrm{Id}_{\mathfrak{g}} \quad ; \quad \alpha^j_i = {\alpha^i_j}^{-1} \quad ; \quad \alpha^k_j \circ \alpha^j_i = \alpha^k_i \tag{2.2.7}$$

Concerning the map χ_{ij}, one has

$$\alpha^i_j \circ \chi_{ji} + \chi_{ij} = 0 \quad ; \quad \chi_{ii} = 0 \quad ; \quad \chi_{kj} + \alpha^k_j \circ \chi_{ij} = \chi_{ik} \tag{2.2.8}$$

The representation of $\Gamma(T\mathcal{M})$ on the maps α^j_i gives the relation

$$(X{\cdot}\alpha^i_j)(\gamma_j) - \alpha^i_j(X{\cdot}\gamma_j) = [\alpha^i_j(\gamma_j), \chi_{ij}(X)] \tag{2.2.9}$$

for any $X \in \Gamma(T\mathcal{U}_{ij})$ and $\gamma_j \in \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$.

Given an atlas of Lie algebroid $(\mathcal{U}_i, S_i)_{i\in I}$, an element $\mathfrak{X}$ of A is defined either as a global object on the Lie algebroid or as a family of pairs $(X \oplus \gamma_i)_{i\in I}$ where $X = \rho(\mathfrak{X})$ and $\gamma_i : \mathcal{U}_i \to \mathfrak{g}$ which fulfills the relation $X \oplus \gamma_i = X \oplus \alpha^i_j(\gamma_j) + \chi_{ij}(X)$. Conversely, given a family of pairs $(X \oplus \gamma_i)_{i\in I}$, if there exist two maps $\alpha^i_j : \mathfrak{g} \to \mathfrak{g}$ and $\chi_{ij} : \Gamma(T\mathcal{M}) \to \mathfrak{g}$ which fulfill the relations (2.2.7),(2.2.8) and (2.2.9), then they are the local trivializations of a global object defined of a Lie algebroid $\mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M})$. One uses the notation s^j_i to designate the transition from $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$ to $\mathsf{TLA}(\mathcal{U}_j, \mathfrak{g})$ defined as $s^j_i(X \oplus \gamma_i) = X \oplus \alpha^j_i(\gamma_i) + \chi_{ji}(X)$.

The set of maps s_i^j should fulfill the compatibility relations $s_i^j = (s_j^i)^{-1}$, $s_i^i = \mathrm{Id}$ and $s_k^l \circ s_k^j \circ s_i^j = s_i^l$.

Over the open set $\mathcal{U}_{ij}$, the $C^\infty(\mathcal{M})$-linear map $\alpha_j^i \in \Omega^1(\mathfrak{g})_{|\mathcal{U}_{ij}} \otimes \mathfrak{g}$ is represented on the Lie algebra $\mathfrak{g}$ as a matrix-valued function $\{(G_{ij})_a^b\} : \mathcal{U}_{ij} \to \mathrm{M}_{n\times n}(\mathbb{R})$ with $a, b = 1, \ldots, n$. Then, the gluing functions α_i^j act on the basis $(E_a)_{a=1,\ldots,n}$ of the Lie algebra $\mathfrak{g}$ as $\alpha_j^i(E_a) = (G_{ij})_a^b E_b$.

One has established the conditions which permit to describe a Lie algebroid in terms of a family of trivial Lie algebroids. In section 3.3, we will see that this system of local trivializations are compatible with the differential structures defined on A.

A last thing to do is to establish the gluing relations between two local trivializations of A in terms of the maps ∇^0 and Ψ. By writing $S_j(X \oplus \gamma_j)$ and $S_i(X \oplus \gamma_i)$ in terms of the triples $(\mathcal{U}_j, \Psi_j, \nabla_j^0)$ and $(\mathcal{U}_i, \Psi_i, \nabla_i^0)$, respectively, and using (2.2.6), one finds:

$$\nabla_j^0 = \nabla_i^0 + \iota \circ \Psi_i \circ \chi_{ij} \quad \text{and} \quad \Psi_j = \alpha_i^j \circ \Psi_i. \tag{2.2.10}$$

Consider a set of elements $(\gamma_i)_{i\in I} \in \Gamma(\mathcal{U}_i \times \mathfrak{g})$ defined on each open set $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{M}$ such that, on any open set $\mathcal{U}_{ij}$, there exist a map α_j^i, which fulfills the relations (2.2.7), such that $\gamma_i = \alpha_j^i \circ \gamma_j$, then, elements $(\gamma_i)_{i\in I}$ are the local trivializations of a global object defined on L.

2.2.3 Example: Atiyah Lie algebroid

A system of local trivializations of an Atiyah Lie algebroid can be related to a system of local trivializations $(\mathcal{U}_i, s_i)_{i\in I}$ of $\mathcal{P}$, where $(s_i)_{i\in I}$ denotes the set of the local cross-sections $s_i : \mathcal{U}_i \to \mathcal{P}|_{\mathcal{U}_i}$ of $\mathcal{P}$. Over the open subset $\mathcal{U}_{ij}$, one recalls that the transition functions are encoded in the maps $g_{ij} : \mathcal{U}_{ij} \to G$ as $s_i = s_j{\cdot}g_{ij}$.

Consider an Atiyah Lie algebroid associated to a principal fiber bundle $\mathcal{P}(\mathcal{M}, G)$. For any $X \in \Gamma(T\mathcal{U}_i)$ and $u = s_i(p){\cdot}g \in \mathcal{P}$, where $p \in \mathcal{U}_i$ and $g \in G$, one defines the map

$$\nabla_i^0 : \Gamma(T\mathcal{U}_i) \to \Gamma_G(\mathcal{P})|_{\mathcal{U}_i} \quad ; \quad (\nabla_{i,X}^0)_u = (T_*R_g{\cdot}s_i^* X)_{s_i(p)\cdot g}. \tag{2.2.11}$$

It is straightforward to show that it results in right-invariant vector fields so that $\nabla_{i,X}^0 \in \Gamma_G(\mathcal{P})$, for any $X \in \Gamma(T\mathcal{U}_i)$. Using direct computations, one shows that ∇_i^0 fulfills the conditions of the section 2.2.1. Moreover, a geometric computation gives the expression of χ_{ij} in terms of g_{ij} as $\chi_{ij} = g_{ij}\mathrm{d}g_{ij}^{-1}$, which corresponds to the local expression of the Maurer-Cartan 1-form. Then, to any $X \in \Gamma(T\mathcal{U}_{ij})$, the two maps $(\nabla_{j,X}^0)$ and $(\nabla_{i,X}^0)$ are related by the formula

$$(\nabla_{j,X}^0) = (\nabla_{i,X}^0) + (\iota \circ (g_{ij}^{-1} X{\cdot}g_{ij}))_u \tag{2.2.12}$$

The second map $\Psi : \Gamma(\mathcal{U}_i \times \mathfrak{g}) \to \Gamma_G(\mathcal{P}_{|\mathcal{U}_i}, \mathfrak{g})$ is also related to the cross-section s_i as

$$(\Psi_i(\gamma_i))(u) = \mathrm{Ad}_{g^{-1}}\gamma_i(p) \tag{2.2.13}$$

for any $\gamma_i \in \Gamma(\mathcal{U}_i \times \mathfrak{g})$ and $u = s_i(p){\cdot}g$. The map $\Psi_i(\gamma_i)$ is a G-equivariant map $\mathcal{P} \to \mathfrak{g}$, compatible with the relations of the section 2.2.1. Over the open set $\mathcal{U}_{ij}$, the map $\alpha_j^i : \mathfrak{g} \to \mathfrak{g}$ is given in terms of g_{ij} as $\alpha_j^i(\gamma_j) = \mathrm{Ad}_{g_{ij}}(\gamma_j)$, for any $\gamma \in \Gamma(\mathcal{U}_j \times \mathfrak{g})$. Then, the algebraic component γ_j of the local trivialization $X \oplus \gamma_j$ transforms over the open set $\mathcal{U}_{ij}$ as

$$\gamma_j = \mathrm{Ad}_{g_{ij}^{-1}} \gamma_i + g_{ij}^{-1}(X \cdot g_{ij}) \tag{2.2.14}$$

With these definitions, one verifies directly that the relations (2.2.7), (2.2.8) and (2.2.9) are fulfilled. In particular, this last relation can be written as

$$X \cdot (g_{ij}^{-1} \gamma_j g_{ij}) = g_{ij}^{-1} \left(X \cdot \gamma_j + [g_{ij} X \cdot g_{ij}^{-1}, \gamma_j] \right) g_{ij}. \tag{2.2.15}$$

These formulaes show that the gluing functions related to an atlas of Lie algebroids generalize the gluing functions g_{ij} given by the theory of fiber bundles. In the general case, the gluing functions are given by an algebraic endomorphism α_i^j on $\mathfrak{g}$ and a geometric 1-form χ_{ji} defined on $\mathcal{U}_{ij}$. In the particular case of the Atiyah Lie algebroids, these two gluing functions become the adjoint action of the gluing function g_{ij} and the local expression of a Maurer-Cartan 1-form, respectively. In this sense, the maps α_i^j and χ_j^i appears to be more fundamental objects. In chapter 4.2, these computations will be used to generalize the gluing relations of local expressions of connection 1-forms.

Chapter 3

Differential structures on transitive Lie algebroids

In the previous chapter, we have stressed that transitive Lie algebroids encode both geometric and algebraic degrees of freedom and correspond, in this sense, in a generalization of the differential geometry on $\mathcal{M}$. One has also established two equivalent descriptions of transitive Lie algebroids, global or local. Then, we can either consider global objects on A or we can look at their local trivializations over the trivial Lie algebroid $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. These two descriptions establish a local isomorphism of vector spaces between $\mathsf{A}_{|\mathcal{U}_i}$ and $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$. This chapter is devoted to the construction of differential structures on transitive Lie algebroids and their local descriptions. In terms of fiber bundles, differential forms are sections of the (multi-)dual fiber bundle $\wedge^\bullet \mathcal{A}^*$. Differential calculus on Lie algebroids are presented in [Mar08]. In this chapter, we use a description of differential forms in terms of $C^\infty(\mathcal{M})$-multilinear antisymmetric applications defined on A with values in an arbitrary space.

In differential geometry, differential forms are defined on $\mathcal{M}$ with values $C^\infty(\mathcal{M})$. In this sense, they involve only geometric degrees of freedom. On this space of differential forms, the Koszul derivative is defined using the representation of vector fields on $C^\infty(\mathcal{M})$ and the Lie bracket structure on $\Gamma(T\mathcal{M})$. Lie algebroids extend the geometry of vector fields by an additional algebraic component, then differential forms defined on A take into account both its geometric and its algebraic degrees of freedom. Differential forms defined on A are valued either in $C^\infty(\mathcal{M})$, or in the kernel L if A is transitive. The set of differential forms with values in L will play an important role in chapter 4.

Locally, the geometric and the algebraic degrees of freedom of A are described separately. Then, locally, any differential operator defined on A should take into account this distinction. Actually, we show that these differential operators can be written as the sum of two differential operators: the first one is the Koszul derivative related to the representation of $\Gamma(T\mathcal{U})$ and the other one is the Chevalley-Eilenberg derivative related to an adapted representation of $\mathfrak{g}$. These derivatives are compatible with the gluing relations associated to changes of trivializations. Thus, it results in a local isomorphism of differential complexes between forms defined on $\mathsf{A}_{|\mathcal{U}}$ and forms defined on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$.

3.1 Representation space

The general scheme associated to the construction of differential forms on A the definition of representations of Lie algebroids on a given space [Mac05]. A representation of Lie algebroids is given by a map defined on A with values in the space of derivations of a vector bundle $\mathcal{E}$ defined over $\mathcal{M}$. This map preserves the Lie bracket on A so that the differential calculus associated to this representaion is nilpotent. On $\mathcal{M}$, the Koszul derivative is associated to the natural representation of vector fields on $C^\infty(\mathcal{M})$. In the context of Lie algebroids, this process is extended to various representations of A.

3.1.1 Representation of Lie algebroids

In addition to a transitive Lie algebroid $\mathsf{A} \xrightarrow{\rho} \Gamma(T\mathcal{M})$, one considers $(\mathcal{E}, \pi, \mathcal{M}, \mathcal{F})$ a vector bundle over $\mathcal{M}$ with typical fiber a vector space $\mathcal{F}$. One denotes by $\mathrm{End}(\mathcal{E})$ the space of endomorphisms of $\mathcal{E}$ and by $\mathrm{Diff}^1(\mathcal{E})$ the set of maps $\{D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})|\ D$ linear, $D(f{\cdot}a) - f{\cdot}D(a) \in \mathrm{End}(\mathcal{E})$ for any $f \in C^\infty(\mathcal{M})$ and any $a \in \Gamma(\mathcal{E})\}$. The symbol map $\sigma : \mathrm{Diff}^1(\mathcal{E}) \to \mathsf{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E}))$ is defined as:

$$\sigma(D)(f\mathrm{d}g)(a) = f{\cdot}(D(g{\cdot}a) - g{\cdot}D(a)) \tag{3.1.1}$$

for any $f, g \in C^\infty(\mathcal{M})$ and $a \in \Gamma(\mathcal{E})$ where d is the Koszul derivative. One sees that $\sigma(D) = 0$ for any $D \in \mathrm{End}(\mathcal{E})$.

With the identification $\mathsf{Hom}_{C^\infty(\mathcal{M})}(T^*\mathcal{M}, \mathrm{End}(\mathcal{E})) \simeq \Gamma(T\mathcal{M} \otimes \mathrm{End}(\mathcal{E}))$, we can inject $\Gamma(T\mathcal{M})$ into $\Gamma(T\mathcal{M}\otimes\mathrm{End}(\mathcal{E}))$ by the relation $\Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})\otimes\mathrm{Id}_\mathcal{E} \in \Gamma(T\mathcal{M}\otimes\mathrm{End}(\mathcal{E}))$. One defines the space $\mathfrak{D}(\mathcal{E}) = \sigma^{-1}(\Gamma(T\mathcal{M}))$ so that one has a transitive Lie algebroid defined by the short exact sequence of $C^\infty(\mathcal{M})$-modules:

$$0 \longrightarrow \Gamma(\mathrm{End}(\mathcal{E})) \xrightarrow{\iota} \mathfrak{D}(\mathcal{E}) \xrightarrow{\sigma} \Gamma(T\mathcal{M}) \longrightarrow 0 \tag{3.1.2}$$

where ι denotes the injection of $\Gamma(\mathrm{End}(\mathcal{E})) \to \sigma^{-1}(0) \subset \Gamma(\mathrm{Diff}^1(\mathcal{E}))$.

A *representation of Lie algebroids* A on a vector bundle $\mathcal{E}$ is a morphism of Lie algebroids $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$. We define the induced morphism of Lie algebroids $\phi_\mathsf{L} : \mathsf{L} \to \Gamma(\mathrm{End}(\mathcal{E}))$ as $\iota \circ \phi_\mathsf{L}(\ell) = \phi(\iota \circ \ell)$, for any $\ell \in \mathsf{L}$. With abuse of notations, the same symbol ι is used in both the Lie algebroids A and $\mathfrak{D}(\mathcal{E})$. This construction is summarized in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L} \arrow[r, "\iota"] \arrow[d, "\phi_\mathsf{L}"'] & \mathsf{A} \arrow[r, "\rho"] \arrow[d, "\phi"] & \Gamma(T\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Gamma(\mathrm{End}(\mathcal{E})) \arrow[r, "\iota"] & \mathfrak{D}(\mathcal{E}) \arrow[r, "\sigma"] & \Gamma(T\mathcal{M}) \arrow[r] & 0 \end{tikzcd} \tag{3.1.3}$$

Given a local trivialization of Lie algebroids $(\mathcal{U}, \nabla^0, \Psi)$, a representation of Lie algebroids $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ can be locally trivialized over the open set $\mathcal{U}$. Then, the representation of $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ on $\mathcal{E}$ is defined through the pull-back of the maps ∇^0 and Ψ as

$$\phi_{\mathrm{loc}} = \phi \circ \nabla^0 + \iota \circ \phi_\mathsf{L} \circ \Psi \qquad \phi_{\mathsf{L},\,\mathrm{loc}} = \phi_\mathsf{L} \circ \Psi. \tag{3.1.4}$$

3.1.2 Representation of A on itself

This first example gives an illustration of the previous construction. We consider the representation of A on itself where the map $\phi : \mathsf{A} \to \mathrm{Diff}^1(\mathsf{A})$ is given by the adjoint action $\phi : \mathsf{A} \to \mathrm{Diff}^1(\mathsf{A})$ with $\phi(\mathfrak{X})(\mathfrak{Y}) = [\mathfrak{X}, \mathfrak{Y}]$ for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$. We now show that the map ϕ is a representation of A.

To do so, we check that the adjoint action on A is correctly valued into $\mathrm{Diff}^1(\mathsf{A})$ and that the map $\sigma \circ \phi$ is compatible with the anchor.

- For any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$ and $f \in C^\infty(\mathcal{M})$, $[\mathfrak{X}, f{\cdot}\mathfrak{Y}] - f{\cdot}[\mathfrak{X}, \mathfrak{Y}] = (\rho(\mathfrak{X}){\cdot}f)\mathfrak{Y}$ is $C^\infty(\mathcal{M})$-linear with respect to $\mathfrak{Y}$, so that $\phi(\mathfrak{X}) \in \mathrm{Diff}^1(\mathsf{A})$.

- For any $f, g \in C^\infty(\mathcal{M})$ and $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, $\sigma(\phi(\mathfrak{X}))(f\mathrm{d}g)(\mathfrak{Y}) = f\cdot([\mathfrak{X}, g\cdot\mathfrak{Y}] - g\cdot[\mathfrak{X}, \mathfrak{Y}]) = -f\cdot(\rho(\mathfrak{X})\cdot g)\mathfrak{Y}$. The contraction of $\rho(\mathfrak{X}) \in \Gamma(T\mathcal{M})$ with $(f\mathrm{d}g) \in \Omega^1(\mathcal{M})$, gives $\rho(\mathfrak{X})(f\mathrm{d}g) = f\cdot(\rho(\mathfrak{X})\cdot g)$. Thus, we read $\sigma \circ \phi = \rho$, which the compatibility relation between ϕ and σ.

The commutating relations (2.1.4) shows that L is an ideal in A. Thus, the restriction of the representation of A on L is also well-defined.

3.1.3 Representation of Atiyah Lie algebroids on an associated vector bundle

On a principal bundle $\mathcal{P}(\mathcal{M}, G)$, the representation of the structure group G is encoded in the definition of an associated vector bundles. Then, Atiyah Lie algebroids can be a represented on a space of sections of an associated vector bundle. In this case, a representation of right-invariant vector fields $\Gamma_G(\mathcal{P})$ is given by a geometric construction.

Let $\mathcal{E}^{\mathcal{P}} = (\mathcal{P} \times \mathcal{F})/G$ be an associated fiber bundle to the principal bundle with typical fiber a space $\mathcal{F}$. The action of G on $\mathcal{F}$ is given by a representation of the structure group $\ell : G \to \mathrm{Aut}(\mathcal{F})$. The space of sections $\Gamma(\mathcal{E}^{\mathcal{P}})$ of this associated fiber bundle is given by the G-equivariant maps $\{s : \mathcal{P} \to \mathcal{F} \,|\, s(u\cdot g) = \ell_{g^{-1}}\cdot s(u) \ \forall\, g \in G\}$. The Atiyah Lie algebroid $\Gamma_G(\mathcal{P}, \mathfrak{g})$ is represented on $\Gamma(\mathcal{E}^{\mathcal{P}})$ by the map $\phi : \Gamma_G(\mathcal{P}, \mathfrak{g}) \to \mathrm{Diff}^1(\mathcal{E}^{\mathcal{P}})$ defined as

$$(\phi(\mathfrak{X})s)(u) = (\mathfrak{X}\cdot s)(u) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} s(\Phi_{\mathfrak{X},t}(u)) \tag{3.1.5}$$

for any $s \in \Gamma(\mathcal{E}^{\mathcal{P}})$ and $u \in \mathcal{P}$, where $\Phi_{\mathfrak{X},t}(u)$ is the flow associated to $\mathfrak{X} \in \Gamma_G(\mathcal{P})$. We make sure that $\phi(\mathfrak{X})s \in \Gamma(\mathcal{E}^{\mathcal{P}})$, for any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$. To do so, we use the fact that $\mathfrak{X}$ is a right-invariant vector field on $\mathcal{P}$ so that we obtain $(\phi(\mathfrak{X})s)(u\cdot g) = g^{-1}(\phi(\mathfrak{X})s)(u)g$, for any $g \in G$. The following results show that the representation of $\Gamma_G(\mathcal{P})$ defines a representation of Lie algebroids.

- For any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$, $s \in \Gamma(\mathcal{E}^{\mathcal{P}})$, $f \in C^\infty(\mathcal{P})$ and $u \in \mathcal{P}$ one has:

  $$\mathfrak{X}\cdot(f\cdot s)(u) - f(u)\cdot(\mathfrak{X}\cdot s)(u) = (\mathfrak{X}\cdot f)(u)s(u). \tag{3.1.6}$$

  The right-hand term is $C^\infty(\mathcal{M})$-linear with respect to s, then $\phi(\mathfrak{X}) \in \mathrm{Diff}^1(\mathcal{E}^{\mathcal{P}})$.

- We denote by $\pi^*f, \pi^*g \in C^\infty(\mathcal{P})$ the pull-back of the functions $f, g \in C^\infty(\mathcal{M})$. For any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$ and $s \in \Gamma(\mathcal{E}^{\mathcal{P}})$, one has:

  $$\begin{aligned}\sigma(\phi(\mathfrak{X}))(f\mathrm{d}g)(s) &= (\pi^*f)\cdot(\mathfrak{X}\cdot(\pi^*g\cdot s) - (\pi^*g)\cdot(\mathfrak{X}\cdot s)) \\ &= (\pi^*f)\cdot(\mathfrak{X}\cdot(\pi^*g))\cdot s\end{aligned}$$

  A direct computation shows that

  $$\mathfrak{X}\cdot(\pi^*g)(u) = g\left(\pi\left(\tfrac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\Phi_{\mathfrak{X},t}(u)\right)\right) = g\left(\tfrac{\mathrm{d}}{\mathrm{d}t}_{|t=0}\pi\left(\Phi_{\mathfrak{X},t}(u)\right)\right) = (T_*\pi\mathfrak{X}\cdot g)(\pi(u))$$

  for any $u \in \mathcal{P}$. Then, for any $f, g \in C^\infty(\mathcal{M})$, one has $\sigma(\phi(\mathfrak{X}))(f\mathrm{d}g) = (T_*\pi f)\cdot(T_*\pi\mathfrak{X}\cdot g)$ with $T_*\pi\mathfrak{X} = \rho(\mathfrak{X})$, so that we obtain $\sigma \circ \phi = \rho$.

3.2 Differential forms on transitive Lie algebroids

From a geometric point view, dual forms and differentials forms are given in terms of sections of the (multi-)cotangent bundle $\mathcal{A}^*$. Here, we do not define the dual bundle $\mathcal{A}^*$. Instead, we define differential forms in terms of $C^\infty(\mathcal{M})$-multilinear antisymmetric maps defined on the $C^\infty(\mathcal{M})$-module A with values in a space of representation space $\mathcal{E} \to \mathcal{M}$. The correspondence between the "fiber" and the "module" points of view are given by the formula

$$\{C^\infty(\mathcal{M})\text{-multilinear } \omega : \wedge^\bullet \mathsf{A} \to \Gamma(\mathcal{E})\} = \Gamma(\wedge^\bullet \mathcal{A}^* \otimes \mathcal{E}) \tag{3.2.1}$$

A differential complex defined on transitive Lie algebroids A is a graded differential algebra over $C^\infty(\mathcal{M})$ of differential forms with values in a representation space $\mathcal{E}$. With respect to this representation, this graded algebra is equipped with a differential operator, denoted as d_ϕ, which increases the degree of forms by 1. In the first subsection, one gives the general scheme which define this differential complex of forms. Then, in the following subsections, we introduce the differential complexes of forms defined on A with values in $C^\infty(\mathcal{M})$ and in L. The differential complexes defined on trivial Lie algebroids will also be detailed. In the trivial case, the geometric and algebraic degrees of freedom of the Lie algebroid are described in distinct spaces so that the representation can be restricted to one space or the other. Then, associated differential operators are written as the sum of a Koszul derivative and a Chevalley-Eilenberg derivative.

3.2.1 Representation space valued differential complex on A

Given a differential complex of forms defined on A with values in a representation space $\mathcal{E}$, we establish the general scheme to define a differential operator of degree $+1$ on this algebra of forms. In [LM12a] is defined a differential complex for forms defined on A with values in $\Gamma(\mathrm{End}(\mathcal{E}))$.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a Lie algebroid (not necessarily transitive) equipped with a representation $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ defined as in 3.1.1. We denote by $\Omega^\bullet(\mathsf{A}, \mathcal{E})$ the differential complex of forms defined on A with values in $\Gamma(\mathcal{E})$. A differential q-form is a $C^\infty(\mathcal{M})$-linear antisymmetric map defined on $\wedge^q \mathsf{A}$ with values in $\Gamma(\mathcal{E})$. As a graded vector space, this differential complex can be written as

$$\Omega^\bullet(\mathsf{A}, \mathcal{E}) = \bigoplus_{q \geq 0} \Omega^q(\mathsf{A}, \mathcal{E}) = \Omega^0(\mathsf{A}, \mathcal{E}) \oplus \Omega^1(\mathsf{A}, \mathcal{E}) \oplus \ldots \oplus \Omega^q(\mathsf{A}, \mathcal{E}) \oplus \ldots \tag{3.2.2}$$

where $\Omega^q(\mathsf{A}, \mathcal{E})$ denotes the space of q-forms on A with values in $\Gamma(\mathcal{E})$. By convention, we take $\Omega^0(\mathsf{A}, \mathcal{E}) = \Gamma(\mathcal{E})$. With respect to the representation ϕ, the differential complex $\Omega^\bullet(\mathsf{A}, \mathcal{E})$ can be equipped with a differential operator $\widehat{\mathrm{d}}_\phi$ which increases the degree of forms by $+1$ as

$$\begin{aligned}(\widehat{\mathrm{d}}_\phi \omega)(\mathfrak{X}_1, \ldots, \mathfrak{X}_{q+1}) = & \sum_{i=1}^{q+1} (-1)^{i+1} \phi(\mathfrak{X}_i) \cdot \omega(\mathfrak{X}_1, \ldots, \overset{i}{\vee}, \ldots, \mathfrak{X}_{q+1}) \\ & + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([\mathfrak{X}_i, \mathfrak{X}_j], X_1, \ldots, \overset{i}{\vee}, \ldots, \overset{j}{\vee}, \ldots, \mathfrak{X}_{q+1})\end{aligned} \tag{3.2.3}$$

for any $\mathfrak{X}_1, \ldots, \mathfrak{X}_{q+1} \in \mathsf{A}$. Since $\phi : \mathsf{A} \to \mathrm{Diff}^1(\mathcal{E})$ is a morphism of Lie algebras, one has $\widehat{\mathrm{d}}_\phi \circ \widehat{\mathrm{d}}_\phi = 0$.

If there exist a multiplicative operation on $\mathcal{E}$, then the differential complex $\Omega^\bullet(\mathsf{A},\mathcal{E})$ forms a graded algebra of differential forms. Indeed, let $\omega \in \Omega^p(\mathsf{A},\mathcal{E})$ and $\eta \in \Omega^q(\mathsf{A},\mathcal{E})$, then the multiplicative operator $\cdot$ on $\Omega^\bullet(\mathsf{A},\mathcal{E})$ is defined as

$$\Omega^{p+q}(\mathsf{A},\mathcal{E}) \ni (\omega{\cdot}\eta)(\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q}) = \frac{1}{p!q!}\,\epsilon^{a^1a^2\dots a^{p+q}}\omega(\mathfrak{X}_{a^1},\dots,\mathfrak{X}_{a^p}){\cdot}\eta(\mathfrak{X}_{a^{p+1}},\dots,\mathfrak{X}_{a^{p+q}}) \tag{3.2.4}$$

for any $\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q} \in \mathsf{A}$ and where $\epsilon^{a^1a^2\dots a^{p+q}}$ denotes a completely antisymmetric tensor with $\epsilon^{12\dots(p+q)} = +1$. A similar formula is obtained by taking ω (or η) a p-form (q-form) defined on A with values in a right-module (left-module) over $\mathcal{E}$ and it results in a graded module structure.

As a $(p+q)$-form defined on A with values in $\Gamma(\mathcal{E})$, the differential operator $\widehat{\mathrm{d}}_\phi$ acts on $\omega{\cdot}\eta$ as:

$$\widehat{\mathrm{d}}_\phi(\omega{\cdot}\eta) = (\widehat{\mathrm{d}}_\phi\omega){\cdot}\eta + (-1)^p\omega{\cdot}(\widehat{\mathrm{d}}_\phi\eta) \tag{3.2.5}$$

This relation indicates that the differential $\widehat{\mathrm{d}}_\phi$ is a *graded differential operator* on $\Omega^\bullet(\mathsf{A},\mathcal{E})$.

3.2.2 Differential complex on A with values in functions

The differential complex of forms defined on A with values in $C^\infty(\mathcal{M})$ has been studied in [Kub98], [Kub99] and [KM03]. Similarly to the Koszul differential, the differential operator $\widehat{\mathrm{d}}$ associated to this differential complex is given by the representation of the vector fields on $\mathcal{M}$ on the space of functions $C^\infty(\mathcal{M})$ and the Lie bracket structure on $\mathcal{M}$.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a Lie algebroid (not necessarily transitive). One denotes by $\Omega^\bullet(\mathsf{A})$ the differential complex of forms defined on A with values in $C^\infty(\mathcal{M})$. As a graded vector space, $\Omega^\bullet(\mathsf{A})$ can be written as

$$\Omega^\bullet(\mathsf{A}) = \bigoplus_{q\geq 0}\Omega^q(\mathsf{A}) = \Omega^0(\mathsf{A}) \oplus \Omega^1(\mathsf{A}) \oplus \dots \oplus \Omega^q(\mathsf{A}) \oplus \dots \tag{3.2.6}$$

where $\Omega^q(\mathsf{A})$ denotes the space of q-forms on A with values in $C^\infty(\mathcal{M})$. By convention, one takes $\Omega^0(\mathsf{A}) = C^\infty(\mathcal{M})$. The representation of A on $C^\infty(\mathcal{M})$ is given by the anchor $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$. Then, vector fields act naturally on the space of functions $C^\infty(\mathcal{M})$ so that the differential complex $\Omega^\bullet(\mathsf{A})$ can be equipped with a differential operator $\widehat{\mathrm{d}}_{\mathsf{A}}$ defined as

$$\begin{aligned}(\widehat{\mathrm{d}}_{\mathsf{A}}\omega)(\mathfrak{X}_1,\dots,\mathfrak{X}_{q+1}) = &\sum_{i=1}^{q+1}(-1)^{i+1}\rho(\mathfrak{X}_i){\cdot}\omega(\mathfrak{X}_1,\dots,\overset{i}{\vee},\dots,\mathfrak{X}_{q+1})\\ &+\sum_{1\leq i<j\leq q+1}(-1)^{i+j}\omega([\mathfrak{X}_i,\mathfrak{X}_j],X_1,\dots,\overset{i}{\vee},\dots,\overset{j}{\vee},\dots,\mathfrak{X}_{q+1})\end{aligned} \tag{3.2.7}$$

for any $\mathfrak{X}_1,\dots,\mathfrak{X}_{q+1} \in \mathsf{A}$. Since $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$ is a morphism of Lie algebra, one has $\widehat{\mathrm{d}}_{\mathsf{A}} \circ \widehat{\mathrm{d}}_{\mathsf{A}} = 0$. The space $\Omega^\bullet(\mathsf{A})$ forms a graded algebra of forms on the Lie algebroid A. Let $\omega \in \Omega^p(\mathsf{A})$ and $\eta \in \Omega^q(\mathsf{A})$, the multiplicative operator $\wedge$ is defined then as

$$(\omega \wedge \eta)(\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q}) = \frac{1}{p!q!}\,\epsilon^{a^1a^2\dots a^{p+q}}\omega(\mathfrak{X}_{a^1},\dots,\mathfrak{X}_{a^p}){\cdot}\eta(\mathfrak{X}_{a^{p+1}},\dots,\mathfrak{X}_{a^{p+q}}) \tag{3.2.8}$$

for any $\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q} \in \mathsf{A}$ and where $\cdot$ denotes the multiplication in $C^\infty(\mathcal{M})$. As a $(p+q)$-form on A, the differential operator $\widehat{\mathrm{d}}_{\mathsf{A}}$ acts on $\omega \wedge \eta$ as:

$$\widehat{\mathrm{d}}_{\mathsf{A}}(\omega \wedge \eta) = (\widehat{\mathrm{d}}_{\mathsf{A}}\omega) \wedge \eta + (-1)^p\omega \wedge (\widehat{\mathrm{d}}_{\mathsf{A}}\eta) \tag{3.2.9}$$

This indicates that $\widehat{\mathrm{d}}_{\mathsf{A}}$ is a graded differential operator on $\Omega^\bullet(\mathsf{A})$.

3.2.3 Differential complex on A with values in L.

The definition of a differential complex on A with values in L involves an additional algebraic element, not present for differential forms with values in $C^\infty(\mathcal{M})$. This element plays an important role in the construction of gauge theories. Here, the target space of these differential forms is the Lie algebroid L so that the representation of A on L is given by the Lie bracket. Moreover, in section 3.3.2, we will see that the trivialization and the gluing relations associated to this space of forms take into account the algebraic structure of L. This differential complex forms a graded Lie algebra where the Lie bracket defined on L is extended to a graded Lie bracket for forms on A with values in L.

Let $\mathsf{A} \overset{\rho}{\to} \mathcal{M}$ be a transitive Lie algebroid with kernel L. One denotes by $\Omega^\bullet(\mathsf{A},\mathsf{L})$ the differential complex of forms defined on A with values in L. As a graded differential complex, it can be written as

$$\Omega^\bullet(\mathsf{A},\mathsf{L}) = \bigoplus_{q\geq 0} \Omega^q(\mathsf{A},\mathsf{L}) = \Omega^0(\mathsf{A},\mathsf{L}) \oplus \Omega^1(\mathsf{A},\mathsf{L}) \oplus \ldots \oplus \Omega^q(\mathsf{A},\mathsf{L}) \oplus \ldots \tag{3.2.10}$$

where $\Omega^q(\mathsf{A},\mathsf{L})$ denotes the space of q-forms on A with values in L. By convention, one takes $\Omega^0(\mathsf{A},\mathsf{L}) = \mathsf{L}$. With respect to the representation of A on L given by the Lie bracket, the differential complex $\Omega^\bullet(\mathsf{A},\mathsf{L})$ can be equipped with a differential operator $\widehat{\mathrm{d}}$ which increases the degree of forms by $+1$. This definition is similar with the Koszul differential for forms defined on $\mathcal{M}$. In particular, on Atiyah Lie algebroids, this differential is equivalent with the differential operator of the de Rham calculus restricted to the space of right-invariant vector fields $\Gamma_G(\mathcal{P})$. For any $\omega \in \Omega^q(\mathsf{A},\mathsf{L})$, the differential operator $\widehat{\mathrm{d}}$ is defined as

$$\begin{aligned}(\widehat{\mathrm{d}}\omega)(\mathfrak{X}_1,\ldots,\mathfrak{X}_{q+1}) =& \sum_{i=1}^{q+1}(-1)^{i+1}[\mathfrak{X}_i, \omega(\mathfrak{X}_1,\ldots,\overset{i}{\vee},\ldots,\mathfrak{X}_{q+1})] \\ &+ \sum_{1\leq i<j\leq q+1} (-1)^{i+j}\omega([\mathfrak{X}_i,\mathfrak{X}_j], X_1,\ldots,\overset{i}{\vee},\ldots,\overset{j}{\vee},\ldots,\mathfrak{X}_{q+1})\end{aligned} \tag{3.2.11}$$

for any $\mathfrak{X}_1,\ldots,\mathfrak{X}_{q+1} \in \mathsf{A}$. The adjoint representation of L on A is a morphism of Lie algebra so that the one has $\widehat{\mathrm{d}} \circ \widehat{\mathrm{d}} = 0$. The space $\Omega^\bullet(\mathsf{A},\mathsf{L})$ forms a graded Lie algebra. Let $\omega \in \Omega^p(\mathsf{A},\mathsf{L})$ and $\eta \in \Omega^q(\mathsf{A},\mathsf{L})$, the Lie bracket defined on L is extended to forms with values in L so that the element $[\omega,\eta] \in \Omega^{p+q}(\mathsf{A},\mathsf{L})$ is defined by the relation

$$[\omega,\eta](\mathfrak{X}_1,\ldots,\mathfrak{X}_{p+q}) = \frac{1}{p!q!}\epsilon^{a^1a^2\ldots a^{p+q}}[\omega(\mathfrak{X}_{a^1},\ldots,\mathfrak{X}_{a^p}),\eta(\mathfrak{X}_{a^{p+1}},\ldots,\mathfrak{X}_{a^{p+q}})] \tag{3.2.12}$$

for any $\mathfrak{X}_1,\ldots,\mathfrak{X}_{p+q} \in \mathsf{A}$. Form this definition, one proves following relations. For any $\omega_1 \in \Omega^{k_1}(\mathsf{A},\mathsf{L}), \omega_2 \in \Omega^{k_2}(\mathsf{A},\mathsf{L})$ and $\omega_3 \in \Omega^{k_3}(\mathsf{A},\mathsf{L})$, the graded Lie bracket on $\Omega^\bullet(\mathsf{A},\mathsf{L})$ fulfill the following relations

- $[\omega_1,\omega_2] = (-1)^{k_1\cdot k_2+1}[\omega_2,\omega_1]$
- $[\omega_1,[\omega_2,\omega_3]] = [[\omega_1,\omega_2],\omega_3] + (-1)^{k_1\cdot k_2}[\omega_2,[\omega_1,\omega_3]]$.

These relations prove that the Lie bracket defined on the differential complex of forms is a *graded Lie bracket* on $\Omega^\bullet(\mathsf{A},\mathsf{L})$. As a differential form of degrees $p+q$ defined on A with values in L, the differential operator $\widehat{\mathrm{d}}$ acts on $[\omega,\eta]$ as $\widehat{\mathrm{d}}[\omega,\eta] = [\widehat{\mathrm{d}}\omega,\eta] + (-1)^p[\omega,\widehat{\mathrm{d}}\eta]$. This indicates that $\widehat{\mathrm{d}}$ is a graded differential operator on $\omega^\bullet(\mathsf{A},\mathsf{L})$.

The differential complex $\Omega^\bullet(\mathsf{A},\mathsf{L})$ is a left-module over $C^\infty(\mathcal{M})$ and this it is also a left-module over the differential complex $\Omega^\bullet(\mathsf{A})$. Let $\omega \in \Omega^p(\mathsf{A})$ and $\eta \in \Omega^q(\mathsf{A},\mathsf{L})$, the left action of $\Omega^\bullet(\mathsf{A})$ on $\Omega^\bullet(\mathsf{A},\mathsf{L})$ is given by the relation

$$\Omega^{p+q}(\mathsf{A},\mathsf{L}) \ni (\omega{\cdot}\eta)(\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q}) = \frac{1}{p!q!}\,\epsilon^{a^1a^2\dots a^{p+q}}\omega(\mathfrak{X}_{a^1},\dots,\mathfrak{X}_{a^p}){\cdot}\eta(\mathfrak{X}_{a^{p+1}},\dots,\mathfrak{X}_{a^{p+q}}) \tag{3.2.13}$$

for any $\mathfrak{X}_1,\dots,\mathfrak{X}_{p+q} \in \mathsf{A}$. As a $(p+q)$-form defined on A with values in L, the differential operator $\widehat{\mathrm{d}}$ acts on $[\omega,\eta]$ as: $\widehat{\mathrm{d}}(\omega{\cdot}\eta) = (\widehat{\mathrm{d}}_{\mathsf{A}}\omega{\cdot}\eta + (-1)^p(\omega{\cdot}\widehat{\mathrm{d}}\eta)$. This indicates that $\widehat{\mathrm{d}}$ is a graded differential operator on $\omega^\bullet(\mathsf{A},\mathsf{L})$.

3.2.4 Cartan operation on $\Omega^\bullet(\mathsf{A},\mathsf{L})$

A Cartan operation ([Car50b; Car50a]) defined on a differential complex permits to distinguish different subspaces such as the horizontal subspace, the invariant subspace and the basic subspace. Here, we consider only a Cartan operation on $\Omega^\bullet(\mathsf{A},\mathsf{L})$. This operation is defined by the data of an *inner operation* i, a differential operator which decreases the degree of forms by 1, and a Lie derivative. Initially, the Lie derivative was defined as a geometric operation defined on vector fields of $\mathcal{M}$ and on the dual space $\Gamma(T^*\mathcal{M})$. Here, we give an algebraic definition of the Lie derivative which uses both the differential operator $\widehat{\mathrm{d}}$ and the inner operation i.

We consider L as a totally intransitive Lie agebroid. Then, the Cartan operation defines an infinitesimal gauge action of the kernel L on the transitive Lie algebroid A. In section (6.2.1), we will see that the Lie derivative of L, applied to connection 1-forms defined on an Atiyah Lie algebroids, gives exactly the infinitesimal version of the action of the gauge group given by the space of vertical automorphisms on $\mathcal{P}$. This correspondence leads to consider the Lie derivative along L as the *infinitesimal geometric gauge action* for gauge theories defined on transitive Lie algebroids.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a transitive Lie algebroid over $\mathcal{M}$ equipped with the differential complex $(\Omega^\bullet(\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$ and let $\mathsf{L} \xrightarrow{0} \mathcal{M}$ be the totally intransitive Lie algebroid corresponding to the kernel of A. The *Cartan operation* of L on the differential complex $(\Omega^\bullet(\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$ is defined by the *inner operation* $i_\ell : \Omega^q(\mathsf{A},\mathsf{L}) \to \Omega^{q-1}(\mathsf{A},\mathsf{L})$ and by a Lie derivative $\mathrm{L}_\ell : \Omega^q(\mathsf{A},\mathsf{L}) \to \Omega^q(\mathsf{A},\mathsf{L})$, for any $\ell \in \mathsf{L}$.

- The inner operation i is defined on any $\omega \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ as $(i_\ell\omega)(\mathfrak{X}_1,\dots,\mathfrak{X}_{q-1}) = \omega(\iota\circ\ell,\mathfrak{X}_1,\dots,\mathfrak{X}_{q-1})$, for any $\ell \in \mathsf{L}$ and $\mathfrak{X}_1,\dots,\mathfrak{X}_{q-1} \in \mathsf{A}$. This inner operation is graded differential operator in the sense that $i_\ell(\omega{\cdot}\eta) = (i_\ell\omega){\cdot}\eta + (-1)^q\omega{\cdot}(i_\ell\eta)$ for any $\omega \in \Omega^q(\mathsf{A})$, $\eta \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ and $\ell \in \mathsf{L}$.
- The Lie derivative along L is defined as $\mathrm{L}_\ell = \widehat{\mathrm{d}}\circ i_\ell + i_\ell\circ\widehat{\mathrm{d}}$ for any $\ell \in \mathsf{L}$. It acts on $\Omega^\bullet(\mathsf{A},\mathsf{L})$ as a derivation *i.e.* $\mathrm{L}_\ell(\omega{\cdot}\eta) = (\mathrm{L}_\ell\omega){\cdot}\eta + \omega{\cdot}(\mathrm{L}_\ell\eta)$.

The inner operation and the Lie derivative fulfill the following relations

$$i_{(f\cdot\gamma)} = f{\cdot}i_\gamma \quad ; \quad i_\gamma{\cdot}i_\eta + i_\eta{\cdot}i_\gamma = 0 \quad ; \quad [\mathrm{L}_\gamma, i_\eta] = i_{[\gamma,\eta]} \quad ; \quad [\mathrm{L}_\gamma,\mathrm{L}_\eta] = \mathrm{L}_{[\gamma,\eta]} \tag{3.2.14}$$

for any $\gamma,\eta \in \mathsf{L}$ and $f \in C^\infty(\mathcal{M})$. One denotes $(\mathsf{L},i,\mathrm{L})$ the Cartan operation on $(\Omega^\bullet(\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$ and we use the following terminology

- *Horizontal forms* in $\Omega^\bullet(\mathsf{A},\mathsf{L})$ are elements $\omega \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ such that $i_\ell\omega = 0$, for any $\ell \in \mathsf{L}$. One denotes by $\Omega^\bullet(\mathsf{A},\mathsf{L})_{\mathrm{hor}}$ the space of horizontal forms with respect to the Cartan operation.

- *Invariant forms* in $\Omega^\bullet(\mathsf{A},\mathsf{L})$ are elements $\omega \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ such that $L_\ell\omega = 0$, for any $\ell \in \mathsf{L}$. One denotes by $\Omega^\bullet(\mathsf{A},\mathsf{L})_{\text{inv}}$ the space of invariant forms with respect to the Cartan operation.

- *Basic forms* in $\Omega^\bullet(\mathsf{A},\mathsf{L})$ are elements $\omega \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ such that $i_\ell\omega = 0$ and $L_{\mathsf{L}}\omega = 0$, for any $\ell \in \mathsf{L}$. One denotes by $\Omega^\bullet(\mathsf{A},\mathsf{L})_{\text{basic}}$ the space of basic forms with respect to the Cartan operation.

3.2.5 Differential complex on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ with values in functions

On trivial Lie algebroids, the distinction between the geometric and the algebraic degrees of freedom is also apparent in the study of differential complexes. Indeed, as graded vector spaces, differential complexes can be decomposed with respect to their degrees of forms. Then, homogeneous terms turn out to be decomposed with respect to their geometric degrees of forms and their algebraic degrees of forms. The important point of this decomposition is the bi-graduation of the total complex which permits to decompose any element of this differential complex on a graded tensorial product of basis.

Moreover, the distinction between the two spaces $\Gamma(T\mathcal{M})$ and $\Gamma(\mathcal{M}\times\mathfrak{g})$ induces two separate representations, one for each space. The associated differential operator defines on $\mathsf{TLA}(\mathcal{U},\mathfrak{g})$ inherits this structure. For the space of forms defined on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ with values in $C^\infty(\mathcal{M})$, vector fields defined on $\mathcal{M}$ are represented by their geometric action on the space of functions whereas algebraic elements $\Gamma(\mathcal{M}\times\mathfrak{g})$ are trivially represented on it. It results in the decomposition of the total differential operator δ as the sum of a Koszul derivative associated to the geometric representation of $\Gamma(T\mathcal{M})$ and a Chevalley-Eilenberg derivative.

Let $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ be a trivial Lie algebroid over $\mathcal{M}$ modeled on the Lie algebra $\mathfrak{g}$. One denotes by $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M})$ the differential complex of forms defined on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ with values in $C^\infty(\mathcal{M})$. As a graded vector space, it can be decomposed as

$$\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}) = \bigoplus_{q=0} \Omega^q_{\mathsf{TLA}}(\mathcal{M}) = \Omega^0_{\mathsf{TLA}}(\mathcal{M}) \oplus \Omega^1_{\mathsf{TLA}}(\mathcal{M}) \oplus \ldots \oplus \Omega^q_{\mathsf{TLA}}(\mathcal{M}) \oplus \ldots \tag{3.2.15}$$

where $\Omega^q_{\mathsf{TLA}}(\mathcal{M})$ denotes the space of q-form on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ with values in $C^\infty(\mathcal{M})$ and, by convention, one takes $\Omega^0_{\mathsf{TLA}}(\mathcal{M}) = C^\infty(\mathcal{M})$. Each of the terms of homogeneous degree have a bi-graduation which comes from the decomposition of $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ into $\Gamma(T\mathcal{M})\oplus\Gamma(\mathcal{M}\times\mathfrak{g})$. Then, this differential complex is also considered as a "trivial" differential complex of Lie algebroids in the sense that geometric degrees of forms and algebraic degrees of forms are separated. The differential complex $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M})$ can be written as

$$\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}) = \Omega^\bullet(\mathcal{M}) \otimes \Omega^\bullet(\mathfrak{g}) \tag{3.2.16}$$

where $\Omega^\bullet(\mathcal{M})$ denotes the space of the de Rham forms, $\Omega^\bullet(\mathfrak{g})$ denotes the space of the Chevalley-Eilenberg forms and the symbol $\otimes$ denotes a graded tensorial product. The Koszul differential acts on $\Omega^\bullet(\mathcal{M})$ and the Chevalley-Eilenberg differential acts on $\Omega^\bullet(\mathfrak{g})$. These two differentials induce the existence of a global differential operator δ, acting on the graded tensorial product of these two spaces. Then, $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M})$ forms a graded tensorial product of differential algebras.

The differential operator δ uses both the representation of vector fields on $C^\infty(\mathcal{M})$ and the trivial representation of the Lie algebra $\mathfrak{g}$. It is defined as:

$$(\delta\omega)(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1})$$
$$+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([X_i \oplus \gamma_i, X_j \oplus \gamma_j], X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \tag{3.2.17}$$

for any $X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1} \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$. Obviously, one has $\delta \circ \delta = 0$.

As it has been already noticed in (3.2.16), the distinction of representations of δ leads to decompose the differential operator δ as the sum of two differential operators

$$\delta = \mathrm{d} + s \tag{3.2.18}$$

where d is the Koszul derivative associated to the representation of vector fields $\Gamma(T\mathcal{M})$ on $C^\infty(\mathcal{M})$ defined on $\Omega_{\mathsf{TLA}}(\mathcal{M})$ as

$$(\mathrm{d}\omega)(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1})$$
$$+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([X_i, X_j] \oplus (X_i \cdot \gamma_j - X_j \cdot \gamma_i), X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \tag{3.2.19}$$

for any $X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1} \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$, and s is the Chevalley-Eilenberg derivative, associated to the trivial representation of $\mathfrak{g}$ on $C^\infty(\mathcal{M})$, defined as

$$(s\omega)(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) =$$
$$\sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega(0 \oplus [\gamma_i, \gamma_j], X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \tag{3.2.20}$$

for any $X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1} \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$. These two differential operators increase the total degree of forms of $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M})$. They act separately on the geometric degrees of forms and the algebraic degrees of forms. With respect to the decomposition of the differential complex along its bi-graduation, one has

$$\begin{cases} \mathrm{d} : \Omega^\bullet(\mathcal{M}) \otimes \Omega^\bullet(\mathfrak{g}) \to \Omega^{\bullet+1}(\mathcal{M}) \otimes \Omega^\bullet(\mathfrak{g}) \\ s : \Omega^\bullet(\mathcal{M}) \otimes \Omega^\bullet(\mathfrak{g}) \to \Omega^\bullet(\mathcal{M}) \otimes \Omega^{\bullet+1}(\mathfrak{g}) \end{cases} \tag{3.2.21}$$

These two differential operators are nilpotent and we show that they commute $\mathrm{d} \circ s + s \circ \mathrm{d} = 0$.

Locally, one uses a local chart of $\mathcal{M}$ to decompose differential forms on the graded tensorial product of elements of the basis of the cotangent bundler $T^*\mathcal{U}$ with elements of the basis of the dual Lie algebra $\mathfrak{g}^*$. We denote by $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$ a basis for the cotangent bundle $T^*\mathcal{U}$ and we denote by $(\theta^1, \theta^2, \dots, \theta^n)$ a basis of the dual Lie algebra $\mathfrak{g}^*$. Straightforwardly, we show that each homogeneous q-form $\omega \in \Omega^q_{\mathsf{TLA}}(\mathcal{U})$ can be written as

$$\omega = \sum_{r+s=q} \omega_{\mu^1\mu^2\dots\mu^r a^1 a^2 \dots a^s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_r} \otimes \theta^{a_1} \wedge \theta^{a_2} \wedge \dots \wedge \theta^{a_s} \tag{3.2.22}$$

where the summation over the repetitive indices is implicit. The multi-index field $\omega_{\mu^1 \dots a^s} \in C^\infty(\mathcal{U})$ is a totally antisymmetric field. With respect to this local decomposition, the action of the differential graded operator $\delta = \mathrm{d} + s$ is summarized in the following relations

$$\begin{array}{l|l|c} \mathrm{d} : f \mapsto \partial_\mu f \mathrm{d}x^\mu & \mathrm{d} : \mathrm{d}x^\mu \mapsto 0 & \mathrm{d} : \theta^a \mapsto 0 \\ s : f \mapsto 0 & s : \mathrm{d}x^\mu \mapsto 0 & s : \theta^a \mapsto -\frac{1}{2} C^a_{bc} \theta^b \wedge \theta^c \end{array} \tag{3.2.23}$$

With these rules, the computation of the differential calculus on $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U})$ is implemented at the level of its local decomposition along a graded tensorial product of basis. In this description, the degrees of freedom of the a q-form ω are all contained in the local components.

3.2.6 Differential complex on $\mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ with values in $\Gamma(\mathcal{M} \times \mathfrak{g})$

One uses the same scheme to define the differential complex of forms defined on $\mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ with values in $\Gamma(\mathcal{M} \times \mathfrak{g})$. Here, similarly to the previous case, the distinction between the geometric and the algebraic degrees of forms occurs explicitly. However, here, $\Gamma_G(\mathcal{P}, \mathfrak{g})$, the target space of these differential forms, supports both the representation of $\Gamma(T\mathcal{M})$ and the adjoint representation of $\mathfrak{g}$ on the $\mathfrak{g}$-component.

On $\Omega^\bullet(\mathsf{A}, \mathsf{L})$, each of the terms of homogeneous degree have a bi-graduation which comes from the decomposition of $\mathsf{TLA}(\mathcal{M}, \mathfrak{g})$ into $\Gamma(T\mathcal{M}) \oplus \Gamma(\mathcal{M} \times \mathfrak{g})$. This bi-graduation leads to the decomposition of $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ as

$$\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}) = \Omega^\bullet(\mathcal{M}) \otimes \Omega^\bullet(\mathfrak{g}) \otimes \mathfrak{g} \tag{3.2.24}$$

where $\Omega^\bullet(\mathcal{M})$ denotes the differential complex of the de Rham forms equipped with the Koszul derivative, $\Omega^s(\mathfrak{g})$ denotes the differential complex of the Chevalley-Eilenberg forms equipped with the Chevalley-Eilenberg derivative and $\mathfrak{g}$ denotes the Lie algebra. The space $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ is a graded Lie algebra with respect to the graded Lie bracket defined in section 3.2.12.

The differential complex $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$ is equipped with a differential operator $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ which increases the degree of forms by 1. This differential operator uses both the representation of vector fields on $\Gamma(\mathcal{M} \times \mathfrak{g})$ and the adjoint representation of the Lie algebra $\mathfrak{g}$ on itself. It is defined as:

$$\begin{aligned} (\widehat{\mathrm{d}}_{\mathsf{TLA}}\omega)(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) = & \\ & \sum_{i=1}^{q+1} (-1)^{i+1} X_i \cdot \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\ & + \sum_{i=1}^{q+1} (-1)^{i+1} [\gamma_i, \omega(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1})] \\ & + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([X_i \oplus \gamma_i, X_j \oplus \gamma_j], X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \end{aligned} \tag{3.2.25}$$

for any $X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1} \in \mathsf{TLA}(\mathcal{M}, \mathfrak{g})$. Obviously, one has $\widehat{\mathrm{d}}_{\mathsf{TLA}} \circ \widehat{\mathrm{d}}_{\mathsf{TLA}} = 0$.

As it has been already noticed in (3.2.16), the differential operator $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ is decomposed as the sum of two differential operators as:

$$\widehat{\mathrm{d}}_{\mathsf{TLA}} = \mathrm{d} + s' \tag{3.2.26}$$

where d is the Koszul derivative associated to the representation of vector fields $\Gamma(T\mathcal{M})$ on $\Gamma(\mathcal{M}\times\mathfrak{g})$ defined on $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M},\mathfrak{g})$ as in (3.2.19) and s' is the Chevalley-Eilenberg derivative associated to the adjoint representation of $\mathfrak{g}$ on $\Gamma(\mathcal{M}\times\mathfrak{g})$ defined as

$$\begin{aligned}(s'\omega)(X_1\oplus\gamma_1,\dots,X_{q+1}\oplus\gamma_{q+1}) &= \sum_{i=1}^{q+1}(-1)^{i+1}[\gamma_i,\omega(X_1\oplus\gamma_1,\dots,\overset{i}{\vee},\dots,X_{q+1}\oplus\gamma_{q+1})] \\ &\sum_{1\leq i<j\leq q+1}(-1)^{i+j}\omega([\gamma_i,\gamma_j],X_1\oplus\gamma_1,\dots,\overset{i}{\vee},\dots,\overset{j}{\vee},\dots,X_{q+1}\oplus\gamma_{q+1})\end{aligned} \tag{3.2.27}$$

for any $X_1\oplus\gamma_1,\dots,X_{q+1}\oplus\gamma_{q+1}\in\mathsf{TLA}(\mathcal{M},\mathfrak{g})$. These two differential operators increase the total degree of forms of $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M},\mathfrak{g})$ by acting separately on the geometric degrees of forms and the algebraic degrees of forms. Indeed, with respect to the decomposition of the differential complex along its bi-graduation, one has

$$\begin{cases}\mathrm{d}:\Omega^\bullet(\mathcal{M})\otimes\Omega^\bullet(\mathfrak{g})\otimes\mathfrak{g}\to\Omega^{\bullet+1}(\mathcal{M}\otimes\Omega^\bullet(\mathfrak{g})\otimes\mathfrak{g}\\ s:\Omega^\bullet(\mathcal{M})\otimes\Omega^\bullet(\mathfrak{g})\otimes\mathfrak{g}\to\Omega^\bullet(\mathcal{M})\otimes\Omega^{\bullet+1}(\mathfrak{g})\otimes\mathfrak{g}\end{cases} \tag{3.2.28}$$

These two differential operators are nilpotent and one shows that they anticommute $\mathrm{d}\circ s'+s'\circ\mathrm{d}=0$

With respect to a local chart $(\mathcal{U},\varphi)$ of $\mathcal{M}$. Each homogeneous q-form $\omega\in\Omega^q_{\mathsf{TLA}}(\mathcal{U},\mathfrak{g})$ can be written as

$$\omega=\sum_{r+s=q}\omega^a_{\mu^1\mu^2\dots\mu^r a^1a^2\dots a^s}\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\ldots\wedge\mathrm{d}x^{\mu_r}\otimes\theta^{a_1}\wedge\theta^{a_2}\wedge\ldots\wedge\theta^{a_s}\otimes E_a \tag{3.2.29}$$

where the summation over the repetitive indices is implicit. The action of the differential graded operator $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ is summarized in the following relations

$$\begin{array}{l|l|c|l}\mathrm{d}:f\mapsto\partial_\mu f\mathrm{d}x^\mu & \mathrm{d}:\mathrm{d}x^\mu\mapsto 0 & \mathrm{d}:\theta^a\mapsto 0 & \mathrm{d}:E_a\mapsto 0\\ s':f\mapsto 0 & s':\mathrm{d}x^\mu\mapsto 0 & s':\theta^a\mapsto -\frac{1}{2}C^a_{bc}\theta^b\wedge\theta^c & s':E_a\mapsto\theta^b[E_b,E_a]\end{array} \tag{3.2.30}$$

With these rules, the computation of the differential complex $\Omega_{\mathsf{TLA}}(\mathcal{U},\mathfrak{g})$ is implemented at the level of its local decomposition along a graded tensorial product of basis. In this description, the degrees of freedom of the a q-form ω are all contained in the local components.

3.3 Local trivializations of differential complexes

By using an atlas of Lie algebroids $(\mathcal{U}_i,\nabla^0_i,\Psi_i)_{i\in I}$, we locally trivialize differential complexes defined on A to differential complexes defined on $\mathsf{TLA}(\mathcal{U},\mathfrak{g})$. In the previous sections, one has seen that differential complexes defined on $\mathsf{TLA}(\mathcal{M},\mathfrak{g})$ make apparent the bi-graduation of forms associated to the separation between $\Gamma(T\mathcal{M})$ and $\Gamma(\mathcal{M}\times\mathfrak{g})$. This separation gives rise to two differential operators: the Koszul derivative and the Chevalley-Eilenberg derivative. Moreover,with respect to a local chart, elements of the differential complexes are decomposed on the graded tensorial product of basis of $T^*\mathcal{M}$, $\mathfrak{g}^*$ and $\mathfrak{g}$.

This section shows how a differential complex defined on A with values either in $C^\infty(\mathcal{M})$, or in L, is locally trivialized to a differential complex defined on $\mathsf{TLA}(\mathcal{U},\mathfrak{g})$ with values either in $C^\infty(\mathcal{M})$ or in $\Gamma(\mathcal{U}\times\mathfrak{g})$, respectively. In both situations, we will see that this local trivialization realizes a local isomorphism of differential complexes (this local isomorphism has been depicted in [FLM13]).

The local trivialization of differential complexes on A is an important point in the construction of gauge theories as physical models. In the spirit of the YM theories, we define a differential complex defined on a transitive Lie algebroid equipped with a gauge action. Since the "physics is on $\mathcal{M}$", the gauge theory has to be locally trivialized. As local trivializations of global objects, their geometric and algebraic properties inherite from an upper, global, structure.

3.3.1 Local trivialization of forms on A with values in functions

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ a Lie algebroid (not necessarily transitive) equipped with an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I} = (\mathcal{U}_i, \nabla_i^0, \Psi_i)_{i\in I}$. Any element $\mathfrak{X}$ of A can be locally trivialized over the open set $\mathcal{U}_i$ as $X \oplus \gamma_i$. Then, the *local trivialization of the differential complex* $(\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}})$ over $\mathcal{U}_i$ is defined as the pull-back action of the map S_i on elements of $\Omega^\bullet(\mathsf{A})$. Indeed, as S_i maps elements of $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$ to elements of $\mathsf{A}|_{\mathcal{U}_i}$, the pull-back action of S_i maps the differential complexes in the opposite direction.

$$\begin{array}{ccc} \mathsf{A}|_{\mathcal{U}_i} & & \Omega^\bullet(\mathsf{A})|_{\mathcal{U}_i} \\ \uparrow{\scriptstyle S_i} & & \downarrow{\scriptstyle S_i^*} \\ \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) & & \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i) \end{array}$$

Then, the local trivialization of $\omega \in \Omega^q(\mathsf{A})$ is given by the relation

$$\omega_{\mathrm{loc},\, i} = S_i^*\omega \quad ; \quad \omega_{\mathrm{loc},\, i}(X_1 \oplus \gamma_{i,\,1}, \dots, X_q \oplus \gamma_{i,\,q}) = \omega(S_i(X_1 \oplus \gamma_{i,\,1}), \dots, S_i(X_q \oplus \gamma_{i,\,q})) \tag{3.3.1}$$

for any $X_1 \oplus \gamma_{i,\,1}, \dots, X_q \oplus \gamma_{i,\,q} \in \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$. We denote by ω_{ij} the restriction of the q-form $\omega \in \Omega^q(\mathsf{A})$ to the open set $\mathcal{U}_{ij}$. Over the open subset $\mathcal{U}_{ij}$, this q-form can be locally trivialized either as $\omega_{\mathrm{loc},\, i} = \omega \circ S_i$, with respect to the local trivialization $(\mathcal{U}_i, S_i)$, or locally trivialized as $\omega_{\mathrm{loc},\, j} = \omega \circ S_j$, with respect to the local trivialization $(\mathcal{U}_j, S_j)$. These two objects are related by the formula $\omega_{\mathrm{loc},\, i} = \omega_{\mathrm{loc},\, j} \circ s_i^j$ where the map $s_i^j : \mathsf{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \to \mathsf{TLA}(\mathcal{U}_{ij}, \mathfrak{g})$ is the morphism of Lie algebroids defined as in 2.2.2.

With an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I}$, any element ω of the differential complex $\Omega^\bullet(\mathsf{A})$ is locally described on the base manifold as a family of elements $(\omega_{\mathrm{loc},\, i})_{i\in I} \in \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$. Locally, each of map $(S_i)_{i\in I} : \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \to \mathsf{A}|_{\mathcal{U}_i}$ establishes a local isomorphism of vector spaces between $\Omega^\bullet(\mathsf{A})|_{\mathcal{U}_i}$ and $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i)$.

Conversely, consider a family of q-forms $(\eta_i)_{i\in I} \in \Omega_{\mathsf{TLA}}(\mathcal{U}_i)$ defined on each open set $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{M}$ and a set of isomorphisms $s_i^j : \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \to \mathsf{TLA}(\mathcal{U}_j)$ such that on any open set $\mathcal{U}_{ij}$, the forms η_i and η_j are related as $\eta_i = \eta_j \circ s_i^j$. Then, the family of differential q-forms $(\eta_i)_{i\in I}$ are the local trivializations of a global form $\eta \in \Omega^q(\mathsf{A})$.

With respect to an atlas of manifold $(\mathcal{U}_i, \varphi_i)_{i\in I}$, over the open set $\mathcal{U}_i \subset \mathcal{M}$, we denote by $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$ a basis for the cotangent bundle $T^*\mathcal{U}$ and one denotes by $(\theta^1, \theta^2, \dots, \theta^n)$ a basis of the dual Lie algebra $\mathfrak{g}^*$. The differential complex $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U})$ is locally decomposed with respect to its bi-graduation as in the section 3.2.5.

With respect to this decomposition, all the degrees of freedom of ω defined on A are locally contained in the fields $(\omega_{\mathrm{loc}})_{\mu^1\mu^2\dots\mu^r a^1 a^2\dots a^s}$. The geometric and the algebraic degrees of freedom of ω_{loc} are related to the Greek indices and Latin indices, respectively. The local component of a q-form ω which has only Greek indices is said to be *purely geometric* and the local component which has only Latin indices is said to be *purely algebraic*. For $q > n$, the $(n-q)$-form which factorizes the basis $\theta^1 \wedge \dots \wedge \theta^n$ is said to be the *maximal inner* component of ω. In section 5.2.4, this maximal inner component will play an important role in the definition of an integration over the algebraic component of forms defined on A.

Locally, the space of differential 1-forms defined on A takes into account the separation between the geometric and the algebraic degrees of freedom of $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Over $\mathcal{U}_i$, one decomposes the map S_i into the pair (∇_i^0, Ψ_i) so that the 1-form $\omega \in \Omega^1(\mathsf{A})$ is locally trivialized as

$$\omega_{\mathrm{loc},\, i} = \omega \circ \nabla_i^0 + \omega \circ \iota \circ \Psi_i = a_i + b_i \tag{3.3.2}$$

where we have denoted by $a_i := \omega \circ \nabla_i^0 \in \Omega^1(\mathcal{U}_i,)$ and $b_i := \omega \circ \Psi_i \in \Omega^1(\mathcal{U}_i \times \mathfrak{g})$ the geometric part and the algebraic part of $\omega_{\mathrm{loc},\, i}$, respectively. On the open set $\mathcal{U}_{ij}$, the geometric part a_j and the algebraic part b_j are related to a_i and b_i as:

$$a_j = a_i + b_i \circ \chi_{ij} \quad ; \quad b_j = b_i \circ \alpha_j^i \tag{3.3.3}$$

where $\chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}$ and $\alpha_i^j \in \Omega^1(\mathfrak{g})_{|\mathcal{U}_{ij}} \otimes \mathfrak{g}$ are defined as in the section 2.2.2. In the theory of fiber bundles, the local trivialization of a 1-form defined on a principal bundle $\mathcal{P}$ gives a 1-form $\Omega^1(\mathcal{U})$. This would correspond to the geometric part of the previous decomposition. The second part, the algebraic one, is not defined in the theory of fiber bundles and consists in an algebraic extension which is proper to the context of the algebraic forms on Lie algebroids.

The following computation shows that, locally, the differential complex on A is compatible with the differential complex on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. It results in an local isomorphism of differential complexes between $(\Omega^\bullet(\mathsf{A})|_{\mathcal{U}_i}, \widehat{\mathrm{d}}_{\mathsf{A}})$ and $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i), \delta)$. Actually, the gluing functions α_j^i and χ_{ij} associated to the Lie algebroid atlas $(\mathcal{U}_i, \nabla_i^0, \Psi_i)$ commute with the differential calculus and then, trivializations of $\Omega^\bullet(\mathsf{A})$ are also *glued* from one open set $\mathcal{U}_i$ to the other open $\mathcal{U}_j$. This isomorphism establishes an equivalence between the description of a differential calculus on A and a set a differential on $(\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})_{i\in I}$.

For any $(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the local trivialization of $\widehat{\mathrm{d}}_{\mathsf{A}}\omega \in \Omega^{q+1}(\mathsf{A})$ over an open set $\mathcal{U}$ is

$$\begin{aligned}
(\widehat{\mathrm{d}}_{\mathsf{A}}\omega)_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) &= (\widehat{\mathrm{d}}_{\mathsf{A}}\omega)(S(X_1 \oplus \gamma_1), \dots, S(X_{q+1} \oplus \gamma_{q+1})) \\
&= \sum_i^{q+1} (-1)^{i+1} \rho(S(X_i \oplus \gamma_i)) \cdot \omega(S(X_1 \oplus \gamma_1), \dots, \overset{i}{\vee}, \dots, S(X_{q+1} \oplus \gamma_{q+1})) \\
&+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega([S(X_i \oplus \gamma_i), S(X_j \oplus \gamma_j)], \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, S(X_{q+1} \oplus \gamma_{q+1})) \\
&= \sum_i^{q+1} (-1)^{i+1} X_i \cdot \omega_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
&+ \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega_{\mathrm{loc}}([X_i \oplus \gamma_i, X_j \oplus \gamma_j], \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
&= (\delta\omega_{\mathrm{loc}})(X_1 \oplus \gamma_1, \dots, X_q \oplus \gamma_{q+1})
\end{aligned}$$

So that we obtain

$$(\widehat{\mathrm{d}}_{\mathsf{A}}\omega)_{\mathrm{loc}} = \delta\omega_{\mathrm{loc}} \tag{3.3.4}$$

where ω_{loc} is the local trivialization of ω. Moreover, over the open set $\mathcal{U}_{ij}$, we use the fact that the map s_i^j is an isomorphism of Lie algebras to establish the following relation

$$(\widehat{\mathrm{d}}_{\mathsf{A}}\omega)_{\text{loc},\,i} = (\widehat{\mathrm{d}}_{\mathsf{A}}\omega)_{\text{loc},\,j} \circ s_i^j \tag{3.3.5}$$

The study of the local trivializations of the differential complex $(\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}})$ is summarized in the following commutative diagram

$$\begin{tikzcd}
& (\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}}) \arrow[rr, "\widehat{\mathrm{d}}_{\mathsf{A}}"] \arrow[dl, equal] \arrow[dd, "\mathcal{U}_j"] & & (\Omega^{\bullet+1}(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}}) \arrow[dl, equal] \arrow[dd, "S_j"] \\
(\Omega^\bullet(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}}) \arrow[rr, "\widehat{\mathrm{d}}_{\mathsf{A}}"] \arrow[dd, "S_i"'] & & (\Omega^{\bullet+1}(\mathsf{A}), \widehat{\mathrm{d}}_{\mathsf{A}}) \arrow[dd] & \\
& \prod_{i\in I} (\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i), \delta) \arrow[rr, "\delta"] & & \prod_{i\in I} (\Omega^{\bullet+1}_{\mathsf{TLA}}(\mathcal{U}_i), \delta) \\
\prod_{i\in I} (\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i), \delta) \arrow[rr, "\delta"] \arrow[ur, "s_j^{i\,*}"] & & \prod_{i\in I} (\Omega^{\bullet+1}_{\mathsf{TLA}}(\mathcal{U}_i), \delta) \arrow[ur, "s_j^{i\,*}"'] &
\end{tikzcd}$$

3.3.2 Local trivialization of forms on A with values in the kernel

We repeat the same process in order to locally trivialize the differential complex of forms with values in the kernel L of $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$. Here, the fact that the forms are valued in the algebraic space L modifies slightly the calculi and the expressions of the gluing relations. Similarly with the previous case, one shows that the data of a atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I}$ results in the formulation of a local isomorphism of differential complexes between $(\Omega^\bullet(\mathsf{A},\mathsf{L})|_{\mathcal{U}_i}, \widehat{\mathrm{d}})$ and $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a transitive Lie algebroid with kernel L equipped with a atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I} = (\mathcal{U}_i, \nabla^0_i, \Psi_i)_{i\in I}$ Over the open set $\mathcal{U}_i$, the *local trivialization of the differential complex* $(\Omega^\bullet(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}})$ is defined as follow. To any $\omega \in \Omega^q(\mathsf{A},\mathsf{L})$, one has:

$$\omega_{\text{loc}} = \Psi^{-1} \circ \omega \circ S \tag{3.3.6}$$

where $\Psi : \Gamma(\mathcal{U} \times \mathfrak{g}) \to \mathsf{L}_{\mathcal{U}}$ is the vector bundle isomorphism associated to S. Contrary to the previous case, the form ω is L-valued so that one needs the map Ψ^{-1} in order to get ω_{loc} with values in $\Gamma(\mathcal{U}_i \times \mathfrak{g})$.

Over the open set $\mathcal{U}_{ij}$, any element ω can be either locally trivialized as $\omega_{\text{loc},\,i} = \Psi_i^{-1} \circ \omega \circ S_i$, with respect to the local trivialization $(\mathcal{U}_i, S_i)$, or locally trivialized as $\omega_{\text{loc},\,j} = \Psi_j^{-1} \circ \omega \circ S_j$,with respect to the local trivialization $(\mathcal{U}_j, S_j)$. These two local expressions are related by the formula

$$\omega_{\text{loc},\,i} = (\Psi_i^{-1} \circ \Psi_j) \circ \Psi_j^{-1} \circ \omega_{\text{loc},\,j} \circ S_j \circ (S_j^{-1} \circ S_i) = \widehat{\alpha}_j^i(\omega_{\text{loc},\,j}) \tag{3.3.7}$$

where the map $\widehat{\alpha}_j^i : \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_{ij}, \mathfrak{g}) \to \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_{ij}, \mathfrak{g})$ is defined as

$$\widehat{\alpha}_i^j(\eta) = \alpha_i^j \circ \eta \circ s_j^i \tag{3.3.8}$$

for any $\eta \in \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_{ij})$ where $\alpha^j_i = \Psi_j^{-1} \circ \Psi_i : \Gamma(\mathcal{U}_{ij} \times \mathfrak{g}) \to \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$ and $s^j_i : \mathsf{TLA}(\mathcal{U}_{ij}, \mathfrak{g}) \to \mathsf{TLA}(\mathcal{U}_{ij}, \mathfrak{g})$. Contrary to the local expression of forms with values in $C^\infty(\mathcal{U})$, the gluing relations of the local expressions of forms with values in L take into account the gluing relations of the target space. Thus, both the source space and the target space of ω transform accordingly to the geometry of the Lie algebroid.

Conversely, consider a set of q-forms $(\eta_i)_{i\in I} \in \Omega_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$ defined on each opens $(\mathcal{U}_i)_{i\in I}$ of $\mathcal{M}$ with values in $\mathfrak{g}$ and a family of pairs $(\alpha^j_i, s^j_i)_{i,j\in I}$ defined on the open sets $\mathcal{U}_{ij}$ as defined in section 2.2.2 such that, on any open sets $\mathcal{U}_{ij}$, the form η_i is related to η_j as $\eta_i = \alpha^i_j \circ \eta_j \circ s^j_i$. Then, the family of forms $(\eta_i)_{i\in I}$ are the corresponding local trivializations of a global form $\eta \in \Omega^q(\mathsf{A}, \mathsf{L})$.

With respect to a local chart of $\mathcal{M}$, we denote by $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$ a basis for the cotangent bundle $T^*\mathcal{U}$, by $(E_1, E_2, \dots, E_n)$ a basis of the Lie algebra $\mathfrak{g}$ and by $(\theta^1, \theta^2, \dots, \theta^n)$ a basis of the dual Lie algebra $\mathfrak{g}^*$. These basis are used to decompose elements of the differential complex $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$ on the graded tensorial product of basis as depicted in the expression (3.2.29). Then, any q-form $\omega \in \Omega^q(\mathsf{A}, \mathsf{L})$ can be locally trivialized as

$$\omega_{\mathrm{loc}} = \sum_{r+s=q} (\omega_{\mathrm{loc}})^a_{\mu^1\mu^2\dots\mu^r a^1 a^2 \dots a^s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_r} \otimes \theta^{a_1} \wedge \theta^{a_2} \wedge \dots \wedge \theta^{a_s} \otimes E_a \quad (3.3.9)$$

where the tensor field $(\omega_{\mathrm{loc}})^a_{\mu^1\mu^2\dots\mu^r a^1 a^2 \dots a^s} \in C^\infty(\mathcal{U})$ is a totally antisymmetric multi-indices field and corresponds to the local components of ω. One uses the same terminology as section 3.3.1 to designate the *purely geometric* component, the *purely algebraic* component and the *maximal inner* form of $\omega \in \Omega^q(\mathsf{A}, \mathsf{L})$.

For 1-forms defined on A with values in L, one defines their geometric part and their algebraic counter part as previously. Let $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$. With respect to the decomposition (2.2.3), the local trivialization of ω can be written as

$$\omega_{\mathrm{loc}} = \Psi^{-1} \circ \omega \circ \nabla^0 + \Psi^{-1} \circ \omega \circ \Psi = A + B \quad (3.3.10)$$

where one has denoted $A = \Psi^{-1} \circ \omega \circ \nabla^0 \in \Omega^1(\mathcal{U}, \mathfrak{g})$ and $B = \Psi^{-1} \circ \omega \circ \Psi \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ the geometric part and the algebraic part of ω_{loc}, respectively. On the open set $\mathcal{U}_{ij}$, the mapsA_j and B_j are related to the maps A_i and B_i as:

$$A_j = \alpha^j_i \circ A_i + \alpha^j_i \circ B_i \circ \chi_{ij} \quad ; \quad B_j = \alpha^j_i \circ B_i \circ \alpha^i_j \quad (3.3.11)$$

where $\chi_{ij} \in \Omega^1(\mathcal{U}_{ij}) \otimes \mathfrak{g}$ and $\alpha^j_i \in \Omega^1(\mathfrak{g})_{|\mathcal{U}_{ij}} \otimes \mathfrak{g}$.

The gluing relations (3.3.8) are compatible with the differential calculus defined on $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g})$. Here, the computations are more cumbersome than in the previous section, due to the presence of the image space $\Gamma(\mathcal{U} \times \mathfrak{g})$ of ω_{loc}. The following computation shows that the gluing functions associated to the atlas of Lie algebroids have the good properties which permit to establish an isomorphism of differential complexes between $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ and $(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{M}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$. For any $(X_1 \oplus \gamma_1, \dots, X_q \oplus \gamma_{q+1}) \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the local trivialization of $\widehat{\mathrm{d}}\omega \in \Omega^{q+1}(\mathsf{A},\mathsf{L})$ over $\mathcal{U}$ gives

$$\begin{aligned}
&(\widehat{\mathrm{d}}\omega)_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
=&\Psi^{-1} \circ (\widehat{\mathrm{d}}\omega)(S(X_1 \oplus \gamma_1), \dots, S(X_{q+1} \oplus \gamma_{q+1})) \\
=&\sum_{i=1}^{q+1} (-1)^{i+1} \Psi^{-1} \circ [S(X_i \oplus \gamma_i)), \omega(S(X_1 \oplus \gamma_1), \dots, \overset{i}{\vee}, \dots, S(X_{q+1} \oplus \gamma_{q+1}))] \\
&\qquad + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \Psi^{-1} \circ \omega([S(X_i \oplus \gamma_i), S(X_j \oplus \gamma_j)], \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots)
\end{aligned}$$

At this point, it is convenient to compute independently the quantity $\Psi^{-1} \circ [S(X \oplus \gamma), \ell]$. Using the relation (2.2.4), one shows that $\Psi^{-1} \circ [S(X \oplus \gamma), \ell] = X{\cdot}(\Psi^{-1}(\ell)) + [\gamma, \Psi^{-1}(\ell)]$ for any $X \in \Gamma(T\mathcal{U})$ and $\ell \in \mathsf{L}$, so that one obtains

$$\begin{aligned}
&(\widehat{\mathrm{d}}\omega)_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, X_q \oplus \gamma_{q+1}) \\
&\quad= \sum_{i=1}^{q+1} (-1)^{i+1} X_i{\cdot}\omega_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
&\qquad + \sum_{i=1}^{q+1} (-1)^{i+1} [\gamma_i, \omega_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, \overset{i}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
&\qquad\quad + \sum_{1 \leq i < j \leq q+1} (-1)^{i+j} \omega_{\mathrm{loc}}([X_i \oplus \gamma_i, X_j \oplus \gamma_j], \dots, \overset{i}{\vee}, \dots, \overset{j}{\vee}, \dots, X_{q+1} \oplus \gamma_{q+1}) \\
&\quad= \widehat{\mathrm{d}}_{\mathsf{TLA}} \omega_{\mathrm{loc}}(X_1 \oplus \gamma_1, \dots, X_q \oplus \gamma_{q+1}).
\end{aligned}$$

Finally

$$(\widehat{\mathrm{d}}\omega)_{\mathrm{loc}} = \widehat{\mathrm{d}}_{\mathsf{TLA}} \omega_{\mathrm{loc}} \tag{3.3.12}$$

for any $\omega \in \Omega^\bullet(\mathsf{A},\mathsf{L})$ where ω_{loc} is the associated local trivialization. It is also straightforward to establish the following relation that the gluing relations and the derivative on $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U},\mathfrak{g})$ commutes so that one has

$$(\widehat{\mathrm{d}}\omega)_{\mathrm{loc},\,i} = \widehat{\alpha}^i_j \circ (\widehat{\mathrm{d}}\omega)_{\mathrm{loc},\,j} \tag{3.3.13}$$

The study of the local trivializations of the differential complex $(\Omega^\bullet(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}})$ is summarized in the following commutative diagram

$$\begin{tikzcd}
& (\Omega^\bullet(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}}) \arrow[rr, "\widehat{\mathrm{d}}"] \arrow[dl, equal] \arrow[dd, "\mathcal{U}_j"] & & (\Omega^{\bullet+1}(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}}) \arrow[dl, equal] \arrow[dd, "S_j"] \\
(\Omega^\bullet(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}}) \arrow[rr, "\widehat{\mathrm{d}}"] \arrow[dd, "S_i"'] & & (\Omega^{\bullet+1}(\mathsf{A},\mathsf{L}), \widehat{\mathrm{d}}) \arrow[dd] & \\
& \prod_{i\in I} (\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i,\mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) \arrow[rr, "\widehat{\mathrm{d}}_{\mathsf{TLA}}"] & & \prod_{i\in I} (\Omega^{\bullet+1}_{\mathsf{TLA}}(\mathcal{U}_i,\mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) \\
\prod_{i\in I} (\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i,\mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) \arrow[ur, "\widehat{\alpha}^j_i"] \arrow[rr, "\widehat{\mathrm{d}}_{\mathsf{TLA}}"] & & \prod_{i\in I} (\Omega^{\bullet+1}_{\mathsf{TLA}}(\mathcal{U}_i,\mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) \arrow[ur, "\widehat{\alpha}^j_i"'] &
\end{tikzcd}$$

Cohomological properties of Lie algebroids have been studied in [CM08b] to study some deformations of the Lie bracket, and in [Fer02] to define characteristic classes. The local isomorphism between the differential complex $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ and $(\Omega_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$ allows to "bring" the computation of the "upper" cohomology classes in terms of local computations. This is done by defining a generalized Čech-de Rham bicomplex, associated to a Mayer-Vietoris sequence (see appendix A for explicits computations). It results that the cohomology of $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ is isomorph to the cohomology of the total complex $(K^\bullet(\mathcal{U}, \mathfrak{F}), D)$ where $K^\bullet(\mathcal{U}, \mathfrak{F})$ is a generalized Čech-de Rham complex.

Chapter 4

Theory of connections on transitive Lie algebroids

As one can see in the construction of YM models, the role of the connections is deeply rooted in the construction of geometric gauge invariant theories. Historically, the first physical gauge theory was the Maxwell theory of electromagnetism. In the language of the theory of the fiber bundles, this theory is based on the existence of a connection on a $U(1)$-principal bundle over $\mathcal{M}_{3,1}$. In differential geometry, connections on a principal bundle are used to define a complementary subspace of the vertical vectors: the so-called horizontal subspace. There, the choice of a horizontal subspace is equivalent to the data of a 1-form of connection defined on $\Gamma(T\mathcal{P})$ with values in the Lie algebra $\mathfrak{g}$ of the structure group. Locally, by using a local trivialization of $\mathcal{P}$, this connection 1-form gives an element A of $\Omega^1(\mathcal{U}, \mathfrak{g})$.

Given a connection, the covariant derivative on $\mathcal{P}$ is defined as the Koszul derivative acting on a representation space and restricted to the horizontal vector fields. Elegantly, the local trivialization of this covariant derivative extends the geometric derivative ∂_μ by a infinitesimal *inner* displacement $A_\mu \in C^\infty(\mathcal{U}) \otimes \mathfrak{g}$. This gives the geometric origin of the minimal coupling between matter fields and gauge bosons. This covariant derivative on $\mathcal{P}$ also defines the curvature of the connection which gives the field strength associated to the gauge bosons A_μ.

Transitive Lie algebroids are considered as an extension of the theory of fiber bundles. To construct connections on transitive Lie algebroids, we keep the spirit of the previous geometric constructions. We have discussed in section 2.1.1 that, contrary to the kernel L, the geometry of vector fields $\Gamma(T\mathcal{M})$ is not canonically identified in A. Then, a connection on A is related to the definition of a "horizontal" subspace of A *i.e.* a subspace which represents the "vector fields" part of A. Geometrically, a connection on A is given by an "arrow" ∇ going from $\Gamma(T\mathcal{M})$ to A *i.e.* in the opposite direction of ρ. Similarly to the decomposition of the vector fields on $\mathcal{P}$ as the sum of vertical and horizontal vector fields, it results in the vectorial splitting of A as $\mathsf{L} \oplus \mathrm{Im}(\nabla)$.

The geometry of ∇ is encoded into the definition of connection 1-forms $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$ called the *ordinary connection 1-forms* associated to ∇. These ordinary connection 1-forms are induced by a purely geometric object and, according to this, the restriction of ω on L gives the constraint $\omega \circ \ell = -\ell$, for any $\ell \in \mathsf{L}$. This is the condition of *normalization* of ω on L.

A connection on a transitive Lie algebroid is defined within the framework of the differential complex so that the previously seen structures (differential, trivialization, Cartan operation, etc) can be applied on it. Moreover, this connection 1-form is also used to define covariant derivatives and curvatures on transitive Lie algebroids. Locally, using a atlas of Lie algebroids, the ordinary connection 1-form is simply written as $A - \theta$ where

$A \in \Omega^1(\mathcal{U}) \otimes \mathfrak{g}$ and $\theta \in \Omega^1(\mathfrak{g}) \otimes \mathfrak{g}$. The former is the geometric part of ω which corresponds to the YM gauge bosons. The algebraic part θ is the identity map on $\mathfrak{g}$ and has no degrees of freedom: it is reminiscent of the normalization of ω on L. On transitive Lie algebroids, it is more convenient to use an algebraic description of a connection as a connection 1-form normalized on L

This algebraic description of connections on A is used to extend the space of *ordinary connections* to the space of *generalized connections*. This extension is obtained from the space of ordinary connections by relaxing the normalization constraint on L. By doing so, we define the space of *generalized connection 1-forms* ϖ on A. Without this constraint, the generalized connection is seen as a connection 1-form equipped with both geometric and algebraic degrees of freedom. The algebraic component of the generalized connection is given by a field $\tau : \mathsf{L} \to \mathsf{L}$ which measures the obstruction for ϖ to be normalized on L. For $\tau = 0$, the generalized connection is normalized on L and then, it describes a geometric object. Given a *background connection* on A, this generalized connection is decomposed, symbolically, as $\omega \oplus \tau$ where ω denotes an *induced ordinary connection* and τ its algebraic component. These results are also presented in [FLM13].

These two aspects, ordinary connections and algebraic fields, are encapsulated in one single object which supports a differential calculus. This leads to the definition of a covariant derivative and a curvature associated to generalized connection on A. These two objects generalize their corresponding geometric definitions. Their local description is also detailed.

4.1 Ordinary connections

A connection on A is geometrically defined as an splitting of $C^\infty(\mathcal{M})$-modules ∇ which maps vector fields on $\mathcal{M}$ to the Lie algebroid A, such that $\rho \circ \nabla = \mathrm{Id}_{\Gamma(T\mathcal{M})}$. For Atiyah Lie algebroids, this application is related to the choice of a horizontal subspace on a principal bundle. The data of a connection A is equivalent with the data of a 1-form ω defined on A with values in L. This corresponds to the algebraic definition of connections on A. The local trivialization of this 1-form shows that the degrees of freedom of connections are only contained in its geometric component. Also, covariant derivatives and curvatures associated to the connections on A are defined either with respect to the geometric map ∇ or with respect to the algebraic 1-form ω. This will give the pure geometric objects of the theory of fiber bundles, as expected.

4.1.1 Splitting of Lie algebroids

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a transitive Lie algebroid with kernel L. A *connection on the transitive Lie algebroid* A is given by a $C^\infty(\mathcal{M})$-linear map $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$ which realizes a splitting on the short exact sequence of $C^\infty(\mathcal{M})$-modules which defines A *i.e.*

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L} \arrow[r, "\iota"] & \mathsf{A} \arrow[r, "\rho"] & \Gamma(T\mathcal{M}) \arrow[r] \arrow[l, bend left, "\nabla"] & 0 \end{tikzcd} \tag{4.1.1}$$

The $C^\infty(\mathcal{M})$-linear map ∇ is compatible with the anchor ρ in the sense that $\rho \circ \nabla_X = X$ for any $X \in \Gamma(T\mathcal{M})$. This implies that ∇ is an injective map and also that no elements of L can be in the image of ∇. Then, the connection ∇ maps vector fields on $\mathcal{M}$ to the quotient vector space A/L. In analogy with the theory of fiber bundles, one calls the image space of ∇ the *horizontal subspace* of A.

The connection also gives rise to the definition of a map $\Theta : \mathsf{A} \to \mathsf{A}$ which extracts the horizontal part of any element on A. This morphism on A is called the *covariant connection* on A associated to the connection ∇. The map Θ is a $C^\infty(\mathcal{M})$-linear map which maps any element $\mathfrak{X}$ into $\mathrm{Im}(\nabla)$ as

$$\Theta : \mathsf{A} \to \mathsf{A} \quad \text{with} \quad \Theta(\mathfrak{X}) = \rho^*\nabla(\mathfrak{X}) = \nabla_{\rho(\mathfrak{X})} \tag{4.1.2}$$

Directly, one shows that $\Theta \circ \Theta = \Theta$ so that Θ is a projector on A. Then, the transitive Lie algebroid A, as a vector space, can be written $\mathsf{A} = \mathrm{Im}(\Theta) \oplus \ker(\Theta)$. Obviously, the map Θ vanishes on L so that $\mathsf{L} \subset \ker(\Theta)$. Moreover, for any $\mathfrak{X} \in \ker(\Theta)$, one shows that $0 = \Theta(\mathfrak{X}) = \nabla_{\rho(\mathfrak{X})} \leftrightarrow \rho(\mathfrak{X}) = 0$ and then $\mathfrak{X} \in \mathsf{L}$. Thus, $\ker(\Theta) = \mathsf{L}$ and we obtain the decomposition

$$\mathsf{A} = \mathrm{Im}(\Theta) \oplus \mathsf{L} \tag{4.1.3}$$

The kernel L represents the vertical part of the Lie algebroid A so that this decomposition is strictly analog to the decomposition of $T\mathcal{P}$ into a horizontal bundle $H\mathcal{P}$ and a vertical bundle $V\mathcal{P}$. Thus, the connection map ∇ establishes the explicit distinction between elements of A which projects to $\Gamma(T\mathcal{M})$ and elements of the kernel L. Note that this distinction is globally defined *i.e.* without using a local trivialization of A. Moreover, this decomposition is not a decomposition of Lie algebras since $\mathrm{Im}(\Theta)$ is not, *a priori*, a Lie algebra. This consideration is related to the definition of the curvature associated to connections on A.

4.1.2 Ordinary connection 1-forms

The connection ∇ is equivalently described as an algebraic object called the ordinary connection 1-form, which is an element of $\Omega^1(\mathsf{A}, \mathsf{L})$. We will show how the geometric aspect of the connection ∇ is reflected on its connection 1-form by a constraint equation.

Let Θ be the covariant connection associated to ∇. For any $\mathfrak{X} \in \mathsf{A}$, we directly see that the difference $\Theta(\mathfrak{X}) - \mathfrak{X}$ is in the kernel of ρ and depends only on $\mathfrak{X}$. Then, this implies that there exist a $C^\infty(\mathcal{M})$-linear map $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$ defined as

$$\iota \circ \omega(\mathfrak{X}) = \Theta(\mathfrak{X}) - \mathfrak{X}. \tag{4.1.4}$$

The 1-form ω is the *connection 1-form* associated to the connection ∇.

$$\begin{tikzcd} 0 \arrow[r] & \mathsf{L} \arrow[r, "\iota"] & \mathsf{A} \arrow[l, bend left, "\omega"] \arrow[r, "\rho"] & \Gamma(T\mathcal{M}) \arrow[l, bend left, "\nabla"] \arrow[r] & 0 \end{tikzcd} \tag{4.1.5}$$

Directly, one shows that the restriction of ω to the subspace L gives $\omega \circ \iota(\ell) = -\ell$ for any $\ell \in \mathsf{L}$. The 1-form ω is a surjective map $\mathsf{A} \to \mathsf{L}$ which is said to be *normalized on* L. The sign " $-$ " is conventional. This normalization is reminiscent of the geometric definition of the connection. Indeed, the connection ∇ only "sees" the vector fields on $\mathcal{M}$ and thus, it says nothing about the inner degrees of freedom of A. Then, the connection 1-form associated to this connection inherits of this specificity under the forme of a constraint equation. The normalization on L "kills" the algebraic degrees of freedom of the connection 1-form. This interpretation is also apparent by looking at local trivializations of connections.

Conversely, one shows that the data of a 1-form defined on A with values in L normalized on L is equivalent to the data of a connection ∇. Any 1-forms are called *ordinary connection 1-forms* on A. Given an ordinary connection ω, we define the map $\Theta : \mathsf{A} \to \mathsf{A}$ as $\Theta(\mathfrak{X}) = \mathfrak{X} + \iota \circ \omega(\mathfrak{X})$. It vanishes on L, then there exist a map $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$ such that

$\Theta = \rho^*\nabla$ and $\rho\circ\nabla = \mathrm{Id}_{\Gamma(T\mathcal{M})}$. It results that the map ∇ is a splitting of $C^\infty(\mathcal{M})$-modules on the sort exact sequence that defines A.

A connection on a transitive Lie algebroids can be seen either with respect to a splitting of $C^\infty(\mathcal{M})$-modules on the short exact sequence that defines A or with respect to the data of a 1-form on A with values in L normalized on L.

Given a connection 1-form ω, the horizontal subspace of A is defined as the kernel of ω. Let $\mathfrak{X} \in \ker(\omega)$, then $\mathfrak{X} = \Theta(\mathfrak{X})$ so that $\ker(\omega) \subset \mathrm{Im}(\Theta)$. Moreover, one computes $\omega(\Theta(\mathfrak{X})) = \Theta(\mathfrak{X}) - \Theta(\Theta(\mathfrak{X})) = 0$ so that $\mathrm{Im}(\Theta) \subset \ker(\omega)$ and then, we obtain $\ker(\omega) = \mathrm{Im}(\Theta)$. The horizontal subspace of A can be defined either as the target space of the covariant connection Θ, or as the kernel of the ordinary connection 1-form ω. This equivalence is also apparent in the theory of connections on principal fiber bundles. It is interesting to remark that ordinary connections define a short exact sequence of $C^\infty(\mathcal{M})$-modules in the "opposite" direction which defines A

$$0 \longleftarrow \mathsf{L} \xleftarrow{\omega} \mathsf{A} \xleftarrow{\nabla} \Gamma(T\mathcal{M}) \longleftarrow 0 \tag{4.1.6}$$

4.1.3 Local trivialization of ordinary connection 1-forms

We consider a connection on A defined by an ordinary connection 1-form ω. As an element of the differential complex $\Omega^\bullet(\mathsf{A},\mathsf{L})$, we use an atlas of Lie algebroids to give a local description of this connection 1-form. This local trivialization of ω will be related to the local description of the usual YM connections on $\mathcal{M}$.

Over $\mathcal{U}_i \subset \mathcal{M}$, the local trivialization of $\omega \in \Omega^1(\mathsf{A},\mathsf{L})$ is defined as in 3.3.2 so that we write

$$\omega_{\mathrm{loc},,i} = \Psi_i^{-1} \circ \omega \circ S_i = A_i + B_i \tag{4.1.7}$$

where $A_i \in \Omega^1(\mathcal{U}_i, \mathfrak{g})$ and $B_i \in \Omega^1(\mathfrak{g},\mathfrak{g})$. The algebraic part B_i of ω is defined as $\Psi_i^{-1} \circ \omega \circ \iota \circ \Psi$. Then, the normalization of ω on L implies that $B_i = -\theta_i$, where θ_i denotes the identity map over $\Gamma(\mathcal{U}_i \times \mathfrak{g})$. Note that the geometric part A_i is not concerned with this normalization. Then, ordinary connections are locally trivialized over $\mathcal{U}_i$ as:

$$\omega_{\mathrm{loc},\,i} = A_i - \theta_i \quad \text{so that} \quad \omega_{\mathrm{loc},\,i}(X \oplus \gamma_i) = A_i(X) - \gamma_i \tag{4.1.8}$$

One sees that the degrees of freedom of ω is carried only by its geometric component A_i. This is a clear illustration of the fact that ω comes from a geometric object and thus, doesn't carry any algebraic degrees of freedom. With respect to our terminology, one says that the ordinary connection ω is a pure geometric object.

In differential geometry, the local expression of connections are identified by the gluing transformations induced by changes of trivializations. On Lie algebroids, the usual gluing functions are generalized to more abstract objects. With respect to the gluing functions α_i^j and χ_{ji}, the geometric and the algebraic parts of ω_{loc} transform on $\mathcal{U}_{ij}$ as

$$A_j = \alpha_i^j \circ A_i + \chi_{ji} \quad ; \quad \theta_j = \theta_i \tag{4.1.9}$$

The gluing relation between the geometric parts A_i and A_j extends the usual transformations of the YM connections. In the case of Atiyah Lie algebroids, we recover the well-known compatibility relations $A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} \mathrm{d} g_{ij}$. The second gluing relation implies that the map θ is globally defined on $\mathcal{M}$. Actually, the family $(\theta_i)_{i\in I}$ describes the local trivializations of Id_{L}. The local maps $(\theta_i)_{i\in I}$ admits an extension to $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$ given by the relation $\theta_i(X \oplus \gamma_i) = \theta_i(\gamma_i) = \gamma_i$, for any $X \oplus \gamma_i \in \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$. Then, over $\mathcal{U}_{ij}$, we obtain the following gluing relation

$$\theta_i(X \oplus \gamma_i) = \alpha^i_j \theta_j(X \oplus \gamma_j) + \chi_{ij}(X). \tag{4.1.10}$$

This shows that the extended maps $(\theta_i)_{i\in I}$, as defined on $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$, do not define a global object on $\mathcal{M}$. In the following, one will specify if the local maps $(\theta_i)_{i\in I}$ are either the local expressions of Id_{L} or denote a family of maps $\mathsf{TLA}(\mathcal{U}, \mathfrak{g}) \to \Gamma(\mathcal{U} \times \mathfrak{g})$.

With respect to the local trivialization of ω, the covariant connection $\Theta : \mathsf{A} \to \mathsf{A}$ associated to the connection ∇ is locally written in terms of the geometric component A as

$$\Theta_{\mathrm{loc},\, i} : \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \to \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \quad ; \quad \Theta_{\mathrm{loc},\, i}(X \oplus \gamma_i) = X \oplus A_i(X). \tag{4.1.11}$$

Then, the elements $X \oplus A(X)$ correspond to the horizontal elements in $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Here, the algebraic component is given by a differential 1-form defined on $\mathcal{U}$. Directly, one checks that $\omega_{loc,\, i} \circ \Theta_{\mathrm{loc},\, i} = 0$.

4.1.4 Covariant derivative on a representation space

In gauge theories, covariant derivatives result in a “minimal coupling term” between gauge bosons A_μ and vector scalar fields. In [KN96a], they are geometrically defined as derivatives of sections α of an associated vector bundle *restricted to the horizontal vector fields* of $\mathcal{P}$. On transitive Lie algebroids, the notions of derivative, representation space and horizontality are well-defined so that the covariant derivative associated to the connection ∇ can be geometrically defined in a similar way.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be equipped with a representation of Lie algebroid $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$, with $\mathcal{E}$ a representation space of A. For any $s \in \Gamma(\mathcal{E})$, the *covariant derivative* associated to the connection $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$ and the representation $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ is a map $\mathcal{D}_\phi : \Gamma(\mathcal{E}) \to \Omega^1(\mathsf{A}, \mathcal{E})$ defined as:

$$(\mathcal{D}_\phi s)(\mathfrak{X}) = (\widehat{\mathrm{d}}_\phi s)(\Theta(\mathfrak{X})), \tag{4.1.12}$$

for any $\mathfrak{X} \in \mathsf{A}$. The covariant derivative restricts to the horizontal elements of A so that, by writing $\Theta = \rho^* \circ \nabla$, one sees that it is a geometric object defined only on $\Gamma(T\mathcal{M})$.

In gauge field theories, covariant derivatives result in a derivation which is extended by an action of a connection ω. This action of ω is interpreted as a minimal coupling with the gauge bosons. Explicitly, for any $\mathfrak{X} \in \mathsf{A}$, the formula (4.1.12) can be written as

$$\mathcal{D}_\phi s(\mathfrak{X}) = \phi(\mathfrak{X} + \iota \circ \omega(\mathfrak{X}))s = \phi(\mathfrak{X})s + \phi_{\mathsf{L}}(\omega(\mathfrak{X}))s \tag{4.1.13}$$

so that a covariant derivative associated to a connection ω can be written under the form

$$\mathcal{D}_\phi s = \widehat{\mathrm{d}}_\phi s + \phi_{\mathsf{L}}(\omega)s, \tag{4.1.14}$$

where ω is the ordinary connection 1-form associated to the connection ∇.We see clearly that the usual derivative of s is extended by the representation of ω. This expression will play an important role in chapter 6 for the construction of gauge invariant quantities.

As in the previous sections, local trivializations of the covariant derivative associated to a connection on A illustrates its geometric dependency. As a 1-form defined on A with values in the derivation of $\Gamma(\mathcal{E})$, the covariant derivative is locally trivialized over the open set $\mathcal{U} \subset \mathcal{M}$ as

$$\mathcal{D}_{\mathrm{loc}} s(X \oplus \gamma) = \phi_{\mathrm{loc}}(X \oplus A(X))s, \tag{4.1.15}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$ and $s \in \Gamma(\mathcal{E})$, where A is the geometric part of ω. Obviously, $\mathcal{D}_{\mathrm{loc}} s(0 \oplus \gamma) = 0$, for any $\gamma \in \Gamma(\mathcal{U} \times \mathfrak{g})$ and $s \in \Gamma(\mathcal{E})$ so that the covariant derivative vanishes on the vertical component of $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Even if the local trivialization of the covariant derivative is defined on $\Gamma(T\mathcal{U}) \oplus \Gamma(\mathcal{U} \times \mathfrak{g})$, it does "see" only the vector fields part and then, it is not concerned with the algebraic component of trivial Lie algebroids.

As a local 1-form defined on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the covariant derivative can be decomposed, with respect to a local chart, on the basis $(\mathrm{d}x^1, \dots, \mathrm{d}x^m)$ of $T^*\mathcal{U}$ and the dual basis $(\theta^1, \dots, \theta^n)$ of the Lie algebra $\mathfrak{g}$. Directly, we show that the covariant derivative is decomposed only on the geometric part $\mathrm{d}x^\mu$ so that we obtain

$$(\widehat{\mathcal{D}}_\phi)_{\mathrm{loc}} = (\widehat{\mathcal{D}}_\phi)_\mu \mathrm{d}x^\mu \tag{4.1.16}$$

where $(\widehat{\mathcal{D}}_\phi)_\mu$ is an element of $\mathfrak{D}(\mathcal{E})$ which is written as $(\widehat{\mathcal{D}}_\phi)_\mu = \partial_\mu + A^a_\mu \, \phi_{\mathsf{L},\,\mathrm{loc}}(E_a)$, where $(E_1, \dots, E_n)$ denotes a basis of $\mathfrak{g}$.

As an illustration, we give the expression of the covariant derivative in the case of the Atiyah Lie algebroid associated to a principal bundle $\mathcal{P}(\mathcal{M}, G)$. The representation space is given by sections on the associated fiber bundle $\mathcal{E}^{\mathcal{P}} = \mathcal{P} \times_\ell \mathcal{F}$ with fiber $\mathcal{F}$. A section $s \in \Gamma(\mathcal{E})$ is given by a G-equivariant map $\mathcal{P} \to \mathcal{F}$. The representation is defined as in 3.1.3 and the expression of the covariant derivative then becomes

$$\mathcal{D}s(\mathfrak{X}) = \mathfrak{X}{\cdot}s + \ell_*(\omega(\mathfrak{X}))s \tag{4.1.17}$$

for any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$, where ℓ_* is the induced left action of $\mathfrak{g}$ on $\Gamma(\mathcal{E})$.

We have seen in section 3.1.3 that local trivializations of an Atiyah Lie algebroid was explicitly given in terms of local cross-sections of $\mathcal{P}$. Over the open set $\mathcal{U} \subset \mathcal{M}$, one denotes by $\sigma : \mathcal{U} \to \mathcal{P}|_{\mathcal{U}}$ a local cross-section of $\mathcal{P}$. Then, any point $u \in \mathcal{P}|_{\mathcal{U}}$ can be written as $u = \sigma(p){\cdot}g$, with $p = \pi(u)$. One defines the local trivialization of the section $s : \mathcal{P} \to \mathcal{F}$ as $s_{\mathrm{loc}} = s \circ \sigma : \mathcal{U} \to \mathcal{F}$ so that one has $(\nabla^0_X{\cdot}s_{\mathrm{loc}})(p) = (X{\cdot}s_{\mathrm{loc}})(p)$, for any $up \in \mathcal{P}$ and $X \in \Gamma(T\mathcal{U})$, and $\phi_{\mathsf{L}} \circ \Psi(A(X))s_{\mathrm{loc}}(p) = \ell_*(A(X))s_{\mathrm{loc}}(p)$. Finally, one obtains

$$\mathcal{D}_{\mathrm{loc}} s_{\mathrm{loc}}(X \oplus \gamma) = X{\cdot}s_{\mathrm{loc}} + \ell_*(A(X))s_{\mathrm{loc}} \tag{4.1.18}$$

Then, local expressions of covariant derivatives defined on an Atiyah Lie algebroid give the usual definition of the covariant derivative on $\mathcal{M}$. Over the open set $\mathcal{U}_{ij}$, one checks that the gluing relations are compatible with these local expression , in the sense that we obtain $\mathcal{D}_{\mathrm{loc},\,i} s_{\mathrm{loc},\,i} = \ell(g_{ij})\mathcal{D}_{\mathrm{loc},\,j} s_{\mathrm{loc},\,j}$.

4.1.5 Curvature associated to an ordinary connection

The curvature associated to a connection plays an essential role in both the geometry of fiber bundles and in the construction of gauge field theories. From a geometric point of view, it gives the measure of the obstruction for the horizontal lift of a closed curve in $\mathcal{M}$ to be a connected path on $\mathcal{P}$. For the point of view of gauge field theories, it is the simplest covariant object which contains first order derivatives in the gauge bosons A_μ. This curvature of a connection gives the field strength associated to these gauge bosons.

On transitive Lie algebroids, the curvature associated to an ordinary connection have three equivalent definitions. The first one considers the curvature as the obstruction for the map ∇ (or Θ) to preserves the Lie bracket from $\Gamma(T\mathcal{M})$ to A. The second one, similarly to the definition given by [KN96a], defines the curvature as the covariant derivative of the connection 1-form ω. The last definition is given by the Cartan structure equation.

Let A be a transitive Lie algebroid over $\mathcal{M}$ and ∇ be a connection on A. The *curvature* associated to this connection is a 2-form R defined on $\Gamma(T\mathcal{M})$ with values in L defined as the obstruction for ∇ to be a morphism of Lie algebras *i.e.*

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \tag{4.1.19}$$

for any $X, Y \in \Gamma(T\mathcal{M})$. The curvature R is also related to the obstruction for the covariant connection Θ associated to ∇ to preserve the Lie bracket on A. One defines the element $F \in \Omega^2(\mathsf{A}, \mathsf{L})$ as

$$F(\mathfrak{X}, \mathfrak{Y}) = [\Theta(\mathfrak{X}), \Theta(\mathfrak{Y})] - \Theta([\mathfrak{X}, \mathfrak{Y}]) \tag{4.1.20}$$

for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$. The curvature R and the 2-form F are related by the pull-back application of the anchor. Indeed, for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, one computes

$$\rho^* R(\mathfrak{X}, \mathfrak{Y}) = R(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) = [\nabla_{\rho(\mathfrak{X})}, \nabla_{\rho(\mathfrak{Y})}] - \nabla_{[\rho(\mathfrak{X}), \rho(\mathfrak{Y})]} = [\rho^*\nabla(\mathfrak{X}), \rho^*\nabla(\mathfrak{Y})] - \rho^*\nabla([\mathfrak{X}, \mathfrak{Y}]) \tag{4.1.21}$$

By definition, $\rho^*\nabla = \Theta$ so that one has $\rho^* R(\mathfrak{X}, \mathfrak{Y}) = [\Theta(\mathfrak{X}), \Theta(\mathfrak{Y})] - \Theta([\mathfrak{X}, \mathfrak{Y}]) = F(\mathfrak{X}, \mathfrak{Y})$. This proves that F and R are related by the formula $F = \rho^* R$. Since it describes the same object, the 2-form F is also called the *curvature* associated to the connection on A.

The curvature F can be defined as the covariant derivative of the connection 1-form ω. For the covariant derivative, we use the differential operator $\widehat{\mathrm{d}} : \Omega^\bullet(\mathsf{A}, \mathsf{L}) \to \Omega^{\bullet+1}(\mathsf{A}, \mathsf{L})$ associated to the adjoint representation of A on L. The covariant derivative associated to this representation is denoted by $\mathcal{D}$ and we have

$$F(\mathfrak{X}, \mathfrak{Y}) = \mathcal{D}\omega(\mathfrak{X}, \mathfrak{Y}) = \widehat{\mathrm{d}}\omega(\Theta(\mathfrak{X}), \Theta(\mathfrak{Y})). \tag{4.1.22}$$

for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$. By taking the explicit expression of the covariant connection Θ in terms of the connection 1-form ω, we obtain the *Cartan structure equation*

$$F = \widehat{\mathrm{d}}\omega + \frac{1}{2}[\omega, \omega] \tag{4.1.23}$$

where $[\cdot, \cdot]$ is the graded Lie bracket on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$.

As an element of the differential complex $\Omega^\bullet(\mathsf{A}, \mathsf{L})$, the 2-form F can be locally trivialized over $\mathcal{U}$ as $F_{\mathrm{loc}} = \Psi^{-1} \circ F \circ S$. To obtain an explicit expression of F_{loc} in terms of the ω_{loc}, one considers the expression (4.1.23) so that we obtain

$$F_{\mathrm{loc}} = \mathrm{d}(A - \theta) + s'(A - \theta) + \frac{1}{2}[A - \theta, A - \theta] \tag{4.1.24}$$

where $A \in \Omega^1(\mathcal{U}, \mathfrak{g})$ is the geometric component of ω_{loc} and θ denotes the identity map on $\Gamma(\mathcal{U} \times \mathfrak{g})$. The expression of F_{loc} is given in terms of the Koszul derivative d and the Chevalley-Eilenberg derivative s', with the adjoint representation on $\mathfrak{g}$, are coming from the local expression of the differential $\widehat{\mathrm{d}}$ on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. Using the calculation rules (3.2.30), we obtain

$$F_{\mathrm{loc}} = \mathrm{d}A + \frac{1}{2}[A, A] \tag{4.1.25}$$

It results that the local expression of the curvature associated to an ordinary connection on A is a purely geometric object. The action of the Chevalley-Eilenberg derivative is counterbalanced by the component "$-\theta$" of the local trivialization of the connection 1-form. Then, the local curvature depends only on the geometric component A of ω. As an element of $\Omega^2_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$, we decompose F on the basis $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$, a basis of the space of covectors $\Gamma(T^*\mathcal{U})$. Then, we obtain

$$F_{\mathrm{loc}} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \quad \text{where} \quad F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]) \tag{4.1.26}$$

where the symbol $\wedge$ denotes the external product on $\Gamma(T\mathcal{U}_i)$. Here, the components in factor of the graded tensorial basis $\mathrm{d}x^\mu \otimes \theta^a$ and $\theta^a \wedge \theta^b$ vanish so that the only component left is the purely geometric component $F_{\mu\nu} \in C^\infty(\mathcal{U}) \otimes \mathfrak{g}$. In field theory, the tensor $F_{\mu\nu}$ gives exactly the field strength associated to the gauge bosons A_μ.

Over the open set $\mathcal{U}_{ij}$, we directly check that the local components $F_{\mu\nu,i}$ are related to the $F_{\mu\nu,j}$'s by the formula $F_{\mu\nu,j} = \alpha_i^j \circ F_{\mu\nu,i}$ for any $i, j = 1, \dots, I$.

4.2 Generalized connections on transitive Lie algebroids

Within the formalism of the transitive Lie algebroids, any *ordinary* connection ∇ is equivalently defined as a geometric application defined on $\Gamma(T\mathcal{M})$ or as an algebraic 1-form of $\Omega^1(\mathsf{A}, \mathsf{L})$. The geometric origin of the map ∇ results in the normalization of ω on L. This normalization is seen as a constraint equation which "kills" the algebraic degrees of freedom of ω. The local description of an ordinary connection 1-form makes apparent that its degrees of freedom are purely geometric. The geometric nature of ω is also apparent in the definition of the covariant derivative and the curvature associated to this connection.

A generalized connection on A is obtained by dropping out the normalization constraint on L. Basically, a generalized connection ϖ is an element of the space $\Omega^1(\mathsf{A}, \mathsf{L})$.Then, this space contains the space of ordinary connections as a subspace. Then, generalized connections on A extend the ordinary connections by a non-geometric element. The non-geometric component of ϖ is quantified by the map $\tau : \mathsf{L} \to \mathsf{L}$ which measures its obstruction to be normalized on L *i.e.* to be an ordinary connection. For $\tau = 0$, the associated generalized connection is geometric. In this sense, τ measures the “algebraic” component of ϖ.

Given a background (ordinary) connection on A, we show that to any generalized connection, we can associate an ordinary connection on A. Contrary to τ, the latter measures the “geometric” component of ϖ. Then, the generalized connection ϖ is decomposed into the sum of an *induced ordinary connection* ω and an algebraic parameter τ. This duality is also apparent in the local trivialization of ϖ.

We adapt the algebraic definition of the covariant derivative and the curvatures of section 4.1.5 and 4.1.4, to the context of the generalized connection. To do so, we substitute the connection 1-form ω by ϖ. In both cases, the algebraic element τ gives additional terms to these objects. Then, the geometric covariant derivative is extended by the representation of the component τ and the curvature is extended by additional terms, whose interpretations are more tricky and require more detailed computations. In order to interpret these new terms, one we need to use some *mixed local basis*. This mixed local basis is an essential feature which permits to decompose the curvature associated to a generalized connection as a sum of three globally defined objects. Locally, these three objects correspond to the three degrees of forms of the curvature.

4.2.1 Decomposition of a generalized connection on A

In this section, we see how, starting from the definition of a generalized connection, we can isolate its algebraic component τ and its geometric component ω.

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be a transitive Lie algebroid over a manifold $\mathcal{M}$, of dimension m, with kernel L, of dimension n. A *generalized connection* over A is defined as a 1-form $\varpi \in \Omega^1(\mathsf{A}, \mathsf{L})$. Obviously, the space of ordinary connection 1-forms is a subspace of the space of generalized connections on A. We define the *reduced kernel endomorphism* $\tau : \mathsf{L} \to \mathsf{L}$ associated to the generalized connection ϖ as

$$\tau(\ell) = \varpi \circ \iota(\ell) + \ell, \tag{4.2.1}$$

for any $\ell \in \mathsf{L}$. The parameter τ is an endomorphism on L which measures the obstruction for ϖ to be normalized on L. Indeed, for $\tau = 0$, one has $\varpi \circ \iota(\ell) = -\ell$, so that ϖ is an ordinary connection.

A generalized connection cannot be defined as a splitting $\Gamma(T\mathcal{M}) \to \mathsf{A}$ on the short exact sequence which defines A. However, for $\tau \neq 0$, the covariant connection $\Theta : \mathsf{A} \to \mathsf{A}$ can be generalized to a map $\widehat{\Theta} : \mathsf{A} \to \mathsf{A}$, defined as $\widehat{\Theta}(\mathfrak{X}) = \mathfrak{X} + \iota \circ \varpi(\mathfrak{X})$, for any $\mathfrak{X} \in \mathsf{A}$. This map is called the *generalized covariant connection* on A. Here, $\widehat{\Theta}$ is not a projector on A. The obstruction for $\widehat{\Theta}$ to be a projector on A is measured by the parameter τ. Indeed, by a direct computation, one shows that

$$\begin{aligned}
\widehat{\Theta}^2(\mathfrak{X}) &= \widehat{\Theta}(\mathfrak{X} + \iota \circ \varpi(\mathfrak{X})) \\
&= \mathfrak{X} + \iota \circ \varpi(\mathfrak{X}) + \iota \circ \varpi(\mathfrak{X} + \iota \circ \varpi(\mathfrak{X})) \\
&= \mathfrak{X} + \iota \circ \varpi(\mathfrak{X}) + \iota \circ \omega(\mathfrak{X}) - \iota \circ \varpi(\mathfrak{X}) - \tau \circ \mathring{\omega}(\mathfrak{X}) + \tau \circ \varpi(\mathfrak{X}) \\
&= \mathfrak{X} + \iota \circ \omega(\mathfrak{X}) - \tau \circ \mathring{\omega}(\mathfrak{X}) + \tau \circ \varpi(\mathfrak{X}) \\
&= \widehat{\Theta}(\mathfrak{X}) + \tau \circ \varpi(\mathfrak{X})
\end{aligned}$$

Then, we obtain $\widehat{\Theta}^2 = \widehat{\Theta} + \tau \circ \varpi$. Only for $\tau = 0$, we recover the usual covariant connection. The generalized covariant connection doesn't vanish on L anymore. Instead, one shows that $\ker(\widehat{\Theta}) = \ker(\tau)$ instead of $\ker(\Theta) = \mathsf{L}$.

Let $\mathring{\omega}$ be an ordinary connection 1-form on A associated to a *background connection* $\mathring{\nabla}$. Given a generalized connection ϖ and its associated reduced kernel endomorphism τ, the 1-form ω defined as

$$\omega = \varpi + \tau(\mathring{\omega}), \tag{4.2.2}$$

is normalized on L. It is called the *induced ordinary connection 1-form* associated to ϖ, with respect to the background connection $\mathring{\omega}$. For $\tau = 0$, one sees that ϖ and its associated induced ordinary connection are the same object. Then, given a background connection on A, any generalized connection ϖ can be decomposed as the sum of an ordinary connection 1-form on A and an algebraic parameter τ. Then, one has the following decomposition

$$\varpi \overset{\mathring{\omega}}{\longleftrightarrow} (\omega, \tau) \tag{4.2.3}$$

This decomposition shows the repartition of the geometric and the algebraic degrees of freedom of ϖ.

With respect to this decomposition, the generalized covariant connection $\widehat{\Theta}$ associated to ϖ can be related to the covariant connection Θ associated to the induced ordinary connection ω. One obtains the explicit generalization of the covariant connection by a non-geometric parameter τ. A direct computation shows that the map $\widehat{\Theta} : \mathsf{A} \to \mathsf{A}$ can be written as

$$\widehat{\Theta} = \Theta - \iota \circ \tau \circ \mathring{\omega}. \tag{4.2.4}$$

For $\tau = 0$, the algebraic extension of Θ vanishes.

Consider that A is equipped with an atlas of Lie algebroids $(\mathcal{U}_i, \Psi_i, \nabla_i^0)_{i\in I}$. As an element of $\Omega^1(\mathsf{A}, \mathsf{L})$, the generalized connection ϖ is locally trivialized over the open set $\mathcal{U}_i$ as $\varpi_{\text{loc},\,i} = \widehat{A}_i + \widehat{B}_i$, where $\widehat{A}_i$ denotes the geometric part of ϖ_{loc} and $\widehat{B}_i$ denote its algebraic part. However, $\widehat{A}_i$ and $\widehat{B}_i$ are not directly identified with the geometric component ω and the algebraic component τ. Indeed, the computation of the local trivialization of ϖ leads to

$$\left\{\begin{array}{lcl} \widehat{A}_i & = & A_i - \tau_{\text{loc},\,i}(\mathring{A}_i) \\ & & \\ \widehat{B}_i & = & -\theta_i + \tau_{\text{loc},\,i} \end{array}\right. \tag{4.2.5}$$

where A_i and $\mathring{A}_i$ denote the geometric components of ω_{loc} and $\mathring{\omega}_{\text{loc}}$, respectively, and $\tau_{\text{loc},\,i}$ denotes the local trivialization of τ on $\mathcal{U}_i$ as $\tau_{\text{loc},\,i} = \Psi_i^{-1} \circ \tau \circ \Psi_i$. The local description of the endomorphism τ on L belongs to the space $C^\infty(\mathcal{U}_i) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$. Note that, for $\tau_{\text{loc}} = 0$, one obtains the local trivialization of an ordinary connection 1-form. On the open set $\mathcal{U}_{ij}$, the local parts $\widetilde{A}_i$ and $\widetilde{B}_i$ of ϖ transform as in (3.3.11). From these transformations, one computes the gluing relations of the geometric components A_i of the induced ordinary connection and the reduced local endomorphism $\tau_{\text{loc},\,i}$. Then, one obtains

$$\left\{\begin{array}{l} A_i = \alpha_j^i \circ A_j + \chi_{ij} \\ \\ \tau_{\text{loc},\,i} = \alpha_j^i \circ \tau_{\text{loc},\,j} \circ \alpha_i^j, \end{array}\right. \tag{4.2.6}$$

where the maps α_j^i and χ_{ij} are defined as in (2.2.2). The gluing transformation of the field A_i has the correct expression with respect to its status of "geometric connection" on A. In particular, this expression is independent of τ. Local expressions of the induced ordinary connection and the reduced kernel endomorphism associated to ϖ are independently glued on $\mathcal{U}_{ij}$.

In the context of the Atiyah Lie algebroid, the gluing functions α_j^i and χ_{ij} are given by the adjoint action $\mathrm{Ad}_{g_{ij}^{-1}}$ and the local Maurer-Cartan forms $g_{ij}^{-1}\widehat{\mathrm{d}}g_{ij}$, respectively. Then, the relations (4.2.6) become $A_i = g_{ij}^{-1} \circ A_j g_{ij} + g_{ij}^{-1}\mathrm{d}g_{ij}$ and $\tau_{\text{loc},\,i}(\gamma_i) = g_{ij}^{-1}\tau_{\text{loc},\,j}(g_{ij}\gamma_i g_{ij}^{-1})g_{ij}$, for any $\gamma_i \in \Gamma(\mathcal{U}_i \times \mathfrak{g})$. These gluing transformations shows that the induced ordinary connection is identified with an Ehresman connection on $\mathcal{P}$. However, the algebraic parameter τ is not related to a geometric object on $\mathcal{P}$. Instead, it would correspond to a morphism on $\Gamma_G(\mathcal{P}, \mathfrak{g})$ and then, would belong to the space $C^\infty(\mathcal{P}) \otimes \mathfrak{g}^* \otimes \mathfrak{g}$.

4.2.2 Covariant derivative associated to a generalized connection

We define the covariant derivative associated to a generalized connection ϖ in the same way as the covariant derivative associated to an ordinary connection on A. Here, the "horizontal" subspace of A is substituted by the image space of $\widehat{\Theta}$. Formula (4.2.4) shows that this horizontal subspace is actually lifted by a vertical element of L depending on τ. The definition of the *generalized* covariant derivative results in an extension of the geometric covariant derivative by τ.

Let $\mathcal{E}$ be a representation space of A with the representation $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$. One denotes by $s \in \Gamma(\mathcal{E})$ a section on the vector bundle $\mathcal{E}$ so that the *covariant derivative associated to a generalized connection* is defined by the map $\Gamma(\mathcal{E}) \to \Omega^1(\mathsf{A}, \mathcal{E})$ as

$$\widehat{\mathcal{D}}_\phi s(\mathfrak{X}) = \mathrm{d}_\phi s(\widehat{\Theta}(\mathfrak{X})), \tag{4.2.7}$$

for any $\mathfrak{X} \in \mathsf{A}$. With respect to the decomposition of $\widehat{\Theta}$, the covariant derivative associated to ϖ is decomposed as $\widehat{\mathcal{D}}_\phi = \mathcal{D}_\phi - \phi_{\mathsf{L}}(\tau \circ \mathring{\omega})$. Explicitly, this generalized covariant connection can be written in terms of the generalized connection ϖ as $\widehat{\mathcal{D}}_\phi = \widehat{\mathrm{d}}_\phi - \phi_{\mathsf{L}}(\varpi)$.

Again, the reduced kernel endomorphism τ measures the obstruction for the generalized covariant derivative $\widehat{\Theta}$ to vanish on L. This is obvious by the formula $\widehat{\mathcal{D}}_\phi s(\iota \circ \ell) = \phi_{\mathsf{L}}(\tau(\ell))s$, for any $s \in \Gamma(\mathcal{E})$ and $\ell \in \mathsf{L}$. Then, $\widehat{\Theta}$ vanishes on the vertical subspace of A if and only if $\tau = 0$ *i.e.* if and only if the generalized connection ϖ is an ordinary connection 1-form.

As a 1-form with values in $\mathfrak{D}(\mathcal{E})$, the covariant derivative associated to a 1-form ϖ can be locally trivialized with respect to a atlas of Lie algebroids $(\mathcal{U}_i, \nabla_i^0, \Psi_i)_{i \in I}$. Over the open set $\mathcal{U}$, the local trivialization of $\widehat{\mathcal{D}}_\phi$ is defined as

$$\begin{aligned}(\widehat{\mathcal{D}}_\phi)_{\text{loc}}(X \oplus \gamma) &= \phi_{\text{loc}}(X \oplus \gamma) - (\phi_{\mathsf{L}})_{\text{loc}}(\varpi_{\text{loc}}(X \oplus \gamma)) \\ &= \phi_{\text{loc}}(X \oplus A(X)) - (\phi_{\mathsf{L}})_{\text{loc}}(\tau_{\text{loc}}(\mathring{\omega}_{\text{loc}}(X \oplus \gamma)))\end{aligned} \tag{4.2.8}$$

for any $X \oplus \gamma \in \mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. Then, the first term corresponds to the local formulation of the covariant derivative associated to ω and the second term establishes a coupling with the parameter τ_{loc}.

As a local 1-form defined on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the generalized covariant derivative can be decomposed with respect to the dual basis $(\mathrm{d}x^1, \dots, \mathrm{d}x^m)$ of $\Gamma(T\mathcal{U})$ and the dual basis $(\theta^1, \dots, \theta^n)$. The covariant derivative is then decomposed as

$$(\widehat{\mathcal{D}}_\phi)_{\text{loc}} = (\widehat{\mathcal{D}}_\phi)_\mu \mathrm{d}x^\mu + (\widehat{\mathcal{D}}_\phi)_a \theta^a \tag{4.2.9}$$

where $(\widehat{\mathcal{D}}_\phi)_\mu$ and $(\widehat{\mathcal{D}}_\phi)_a$ are elements of $\mathfrak{D}(\mathcal{E})$.

Anticipating on the next sections, one writes the local expression of $\widehat{\mathcal{D}}_\phi$ in the *mixed local basis*. In this basis, the existence of the background connection $\mathring{\omega}$ plays an essential role. To obtain the decomposition of $\widehat{\mathcal{D}}_\phi$ on the mixed local basis, we observe that, locally, the background connection 1-form $\mathring{\omega}$ can be written as $\mathring{A} - \theta$ so that the component θ^a can be written $\theta^a = \mathring{A}^a - \mathring{\omega}^a_{\text{loc}}$, where $\mathring{A}^a \in \Omega^1(\mathcal{U})$ and $\mathring{\omega}^a_{\text{loc}} \in \Omega^1_{\mathsf{TLA}}(\mathcal{U})$. Then, the expression (4.2.9) can be written as

$$(\widehat{\mathcal{D}}_\phi)_{\text{loc}} = (\widehat{\mathcal{D}}_\phi)'_\mu \mathrm{d}x^\mu + (\widehat{\mathcal{D}}_\phi)'_a \mathring{\omega}^a_{\text{loc}} \tag{4.2.10}$$

with $(\widehat{\mathcal{D}}_\phi)'_\mu$ and $(\widehat{\mathcal{D}}_\phi)'$ are elements of $\mathfrak{D}(\mathcal{E})$. In this basis, the components can be written as

$$\begin{cases} (\widehat{\mathcal{D}}_\phi)'_\mu = \phi_{\text{loc}}(\partial_\mu) + A^a_\mu \, \phi_{\mathsf{L},\,\text{loc}}(E_a) \\ \\ (\widehat{\mathcal{D}}_\phi)'_a = \tau^b_a \, \phi_{\mathsf{L},\,\text{loc}}(E_b) \end{cases} \tag{4.2.11}$$

The full aspect of the mixed local basis will be detailed in section 5.2.1. It plays an essential role in the definition of the integration over A and also in the construction of a gauge theories on transitive Lie algebroids.

4.2.3 Curvature associated to a generalized connection

Similarly with the theory of ordinary connections on A, the curvature associated to a generalized connection ϖ can be defined either as the obstruction for the generalized covariant connection $\widehat{\Theta}$ to be a morphism of Lie algebras on A or by the Cartan structure equation associated to ϖ. One will show that these two definitions are equivalent. However, the curvature associated to a generalized connection cannot be defined as the generalized covariant derivative of ϖ.

The decomposition of a generalized connection ϖ as the sum of an ordinary connection ω and its associated reduced kernel endomorphism τ induces an extension of the "ordinary" curvature associated to an ordinary connection 1-form. This extension is composed by two new terms which involve an interaction between the connection 1-form ω and the field τ and also a quadratic term which depends only of the algebraic constraint on τ. After some manipulations, the expression of the generalized curvature gives rise to an ordinary curvature on $\mathcal{M}$, a covariant derivative along ω of the algebraic parameter τ and a potential term which measures the obstruction for τ to preserve the Lie bracket on L.

Locally, the use of a background connection on A is an essential feature in order to decompose correctly the generalized curvature along its differential degrees of forms. Then, one shows that the three components associated to this decomposition each defines a global object on A. As parts of a global single object, these three components inherit the property of the generalized curvature, for instance, with respect to the gauge action of L.

Let $\widehat{\Theta}$ be the generalized covariant connection associated to a generalized connection ϖ. The *generalized curvature* $\widehat{F}$ associated to this generalized connection is defined as

$$\widehat{F}(\mathfrak{X},\mathfrak{Y}) = [\widehat{\Theta}(\mathfrak{X}),\widehat{\Theta}(\mathfrak{Y})] - \widehat{\Theta}([\mathfrak{X},\mathfrak{Y}]) \tag{4.2.12}$$

for any $\mathfrak{X},\mathfrak{Y} \in \mathsf{A}$. Here, the curvature measures the obstruction for $\widehat{\Theta}$ to be a morphism of Lie algebras on A. By writing $\widehat{\Theta}$ in terms of the generalized connection 1-form ϖ, one obtains the Cartan structure equation of the curvature $\widehat{F}$:

$$\widehat{F} = \widehat{\mathrm{d}}\varpi + \frac{1}{2}[\varpi,\varpi] \tag{4.2.13}$$

where $[\cdot,\cdot]$ is the graded Lie bracket defined on $\Omega^\bullet(\mathsf{A},\mathsf{L})$. Obviously, for ϖ an ordinary connection on A, we obtain the expression of the ordinary curvature. In this sense, generalized connections correctly generalize geometric curvatures.

For convenience, one considers the Cartan structure equation (4.2.13) associated to ϖ to define $\widehat{F}$. One decomposes ϖ into its geometric component, given by its induced ordinary connection ω, and its algebraic component, given by its reduced kernel endomorphism τ. By using a non-trivial rearrangement of the terms induced by this decomposition, one computes the expression of generalized curvature associated to ϖ, written in terms of ω and τ. This expression will be *a posteriori* justified by the geometric and algebraic interpretations of each of these terms. The curvature $\widehat{F}$ can be written as

$$\widehat{F} = \mathfrak{R} - (\mathfrak{D}\tau)\circ\mathring{\omega} + \mathring{\omega}^* R_\tau \tag{4.2.14}$$

with

$$\begin{cases} \mathfrak{R} := F - \tau \circ \mathring{F} \\ (\mathfrak{D}\tau)\circ\mathring{\omega} := [\Theta, \tau\circ\mathring{\omega}] - \tau\circ[\mathring{\Theta},\mathring{\omega}] \\ \mathring{\omega}^* R_\tau := \frac{1}{2}\big(\tau\circ[\mathring{\omega},\mathring{\omega}] - [\tau\circ\mathring{\omega},\tau\circ\mathring{\omega}]\big) \end{cases} \tag{4.2.15}$$

where F is the curvature of the induced ordinary connection, $\mathring{F}$ is the curvature of the background connection, Θ is the covariant connection associated to the induced ordinary connection and $\mathring{\Theta}$ is the covariant connection associated to the background connection.

This decomposition of $\widehat{F}$ makes apparent the geometric and the algebraic meaning of the curvature associated to a generalized connection. The following comments will be relevant in the upcoming construction of a gauge invariant field theory.

- The first term $\mathfrak{R}$ involves the curvatures of the induced connection ω and of the background connection $\mathring{\omega}$. As we said before, these two objects are geometric objects defined on the space of vector fields $\Gamma(T\mathcal{M})$. Then, there exist a 2-form $\widehat{R}$ defined on vector fields on $\mathcal{M}$ with values in L associated to $\mathfrak{R}$ and given by the relation $\widehat{R}(X, Y) = \mathfrak{R}(\mathfrak{X}, \mathfrak{Y})$ for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$ such that $\rho(\mathfrak{X}) = X$ and $\rho(\mathfrak{Y}) = Y$. Equivalently, $\widehat{R}$ is related to $\mathfrak{R}$ by the pull-back of the anchor ρ so that one has $\mathfrak{R} = \rho^*(\widehat{R}) \in \Omega^2(\mathcal{M}, \mathsf{L})$.

- The second term $(\mathfrak{D}\tau) \circ \mathring{\omega}$ involves the covariant connections Θ and $\mathring{\Theta}$ associated to the induced ordinary connection and the background connection, respectively. Then, they are geometric objects defined on $\mathcal{U}$ which can be defined with respect to the connection ∇ as $\Theta = \rho^*\nabla$ and with respect to the background connection $\mathring{\nabla}$ as $\mathring{\Theta} = \rho^*\mathring{\nabla}$. Here, the map $\mathring{\nabla}$ is the connection associated to $\mathring{\omega}$. For any $\alpha \in \mathrm{End}(\mathsf{L})$, one defines the map $\mathcal{D}\alpha : \Gamma(T\mathcal{U}) \to \mathrm{End}(\mathsf{L})$ as $\mathcal{D}_X\alpha(\ell) = [\nabla_X, \tau(\ell)] - \tau \circ [\mathring{\nabla}_X, \ell]$, for any $\ell \in \mathsf{L}$. Even if the adjoint representation on L gives a derivation, the difference of these two terms gives an endomorphism so that $\mathcal{D}\alpha$ is well-defined. Moreover, a cumbersome computation shows that for any $X, Y \in \Gamma(T\mathcal{M})$, the map $\mathcal{D}$ is related to the curvatures R and $\mathring{R}$ as

$$\mathcal{D}_X\mathcal{D}_Y\tau - \mathcal{D}_Y\mathcal{D}_X\tau - \mathcal{D}_{[X,Y]}\tau = [R(X, Y), \tau] - \tau([\mathring{R}(X, Y), \mathrm{Id}_{\mathsf{L}}]) \tag{4.2.16}$$

  In order to make apparent the purely geometric component of $(\mathfrak{D}\tau) \circ \mathring{\omega}$, one uses the pull-back by the anchor map ρ so that it can be equivalently written as $(\rho^*\mathcal{D}\tau) \circ \mathring{\omega} = [\rho^*\nabla, \tau \circ \mathring{\omega}] - \tau \circ [\rho^*\mathring{\nabla}, \mathring{\omega}]$. Under this form, one sees that this 2-form mixes the purely geometric degrees and the algebraic ones. This point will be relevant in the local decomposition of this term on a mixed local basis.

- The last term $\mathring{\omega}^* R_\tau$ is the obstruction for τ to be an endomorphism of Lie algebras on L. It is called the *algebraic curvature* of the reduced kernel endomorphism with a background connection.

As expected by the nature of generalized connection on A, the generalized curvature associated to ϖ is related to both purely geometric terms and non-geometric ones. In order to take into account its geometric components, one writes the curvature $\widehat{F}$ associated to the generalized connection ϖ under the form

$$\widehat{F} = \rho^*\widehat{R} - (\rho^*\mathcal{D}\tau) \circ \mathring{\omega} + \mathring{\omega}^* R_\tau. \tag{4.2.17}$$

Directly, one reads that the first term gives always zero if one the argument of $\widehat{F}$ is an element of L. If the two arguments of $\widehat{F}$ are in L, then both the first and the second terms vanish. The third term do not vanish depending on its arguments. Instead, it gives zero if and only if the algebraic parameter τ preserves the Lie bracket on L. These three terms are ordered according to their degrees of forms: the first term is a purely geometric term, the second term mixes both the geometric and the algebraic component of A, and the third term is a purely algebraic term. Moreover, one will see in section 6.3, that these three terms are compatible with the infinitesimal algebraic action of L.

As an element of the differential complex $\Omega^\bullet(\mathsf{A}, \mathsf{L})$, the generalized curvature $\widehat{F}$ can be locally trivialized over $\mathcal{U}$ providing an atlas of Lie algebroids $(\mathcal{U}_i, S_i)$. Over any open set $\mathcal{U}$, it is trivialized as $\widehat{F}_{\text{loc}} = \Psi^{-1} \circ \widehat{F} \circ S$ and then, with respect to the previous decomposition of $\widehat{F}$, one obtains

$$\widehat{F}_{\text{loc}} = (\rho^* \widehat{R})_{\text{loc}} - ((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})_{\text{loc}} + (\mathring{\omega}^* R_\tau)_{\text{loc}} \tag{4.2.18}$$

with

$$\begin{cases} (\rho^* \widehat{R})_{\text{loc}} = \mathrm{d}A + \frac{1}{2}[A, A] - \tau_{\text{loc}}(\mathrm{d}\mathring{A} + \frac{1}{2}[\mathring{A}, \mathring{A}]) \in \Omega^2(\mathcal{U}, \mathfrak{g}) \\ ((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})_{\text{loc}} = (\mathrm{d}\tau_{\text{loc}}) \circ \mathring{\omega}_{\text{loc}} + [A, \tau_{\text{loc}} \circ \mathring{\omega}_{\text{loc}}] - \tau_{\text{loc}}([\mathring{A}, \mathring{\omega}_{\text{loc}}]) \in \Omega^1(\mathcal{U}, \text{End}(\mathfrak{g})) \\ (\mathring{\omega}^* R_\tau)_{\text{loc}} = \frac{1}{2}\left(\tau_{\text{loc}}([\mathring{\omega}_{\text{loc}}, \mathring{\omega}_{\text{loc}}]) - [\tau_{\text{loc}} \circ \mathring{\omega}_{\text{loc}}, \tau_{\text{loc}} \circ \mathring{\omega}_{\text{loc}}]\right) \in C^\infty(\mathcal{U}) \otimes \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}. \end{cases} \tag{4.2.19}$$

On $\mathcal{U}_{ij}$, the gluing transformations of the three components $(\rho^* \widehat{R})_{\text{loc}}$, $((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})_{\text{loc}}$ and $(\mathring{\omega}^* R_\tau)_{\text{loc}}$ are computed with the gluing transformations of the local objects A, $\mathring{A}$ and τ_{loc}. The results are obtained in terms of the gluing functions α_i^j and χ_i^j associated to the atlas of Lie algebroids $(\mathcal{U}_i, \nabla_i^0, \Psi_i)$. Straightforwardly, these relations are

$$\begin{cases} (\rho^* \widehat{R})_{\text{loc}, j} = \alpha_i^j (\rho^* \widehat{R})_{\text{loc}, i} \\ ((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})_{\text{loc}, j} = \widehat{\alpha}_i^j ((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})_{\text{loc}, i} \\ (\mathring{\omega}^* R_\tau)_{\text{loc}, j} = \widehat{\alpha}_i^j (\mathring{\omega}^* R_\tau)_{\text{loc}, i} \end{cases} \tag{4.2.20}$$

These results are compatible with the fact that the three components define global objects on A.

As a 2-form defined on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$, the generalized curvature can be decomposed with respect to the basis $(\mathrm{d}x^1, \mathrm{d}x^2, \dots, \mathrm{d}x^m)$ of $\Gamma(T^*\mathcal{U})$, the basis $(E_1, E_2, \dots, E_n)$ for $\mathfrak{g}$ and the basis $(\theta^1, \theta^2, \dots, \theta^n)$ for the dual space $\mathfrak{g}^*$. Considering the previous decomposition of $\widehat{F}$, it seems natural to use the background connection to define the mixed local basis associated to $\mathring{\omega}$. Then, similarly to the local decomposition of the generalized covariant derivative (section 4.2.2), the generalized curvature is decomposed on the basis $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\text{loc}})$ instead of the basis $(\mathrm{d}x^\mu, \theta^a)$. This change of basis of $\Omega_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$ is wider explained in the section 5.2.1. This local description of $\widehat{F}$ in the mixed local basis gives

$$\widehat{F}_{\text{loc}} = \widehat{F}^a_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes E_a + \widehat{F}^b_{\mu a}\, \mathrm{d}x^\mu \wedge \mathring{\omega}^a_{\text{loc}} \otimes E_b + \widehat{F}^c_{ab}\, \mathring{\omega}^a_{\text{loc}} \wedge \mathring{\omega}^b_{\text{loc}} \otimes E_c \tag{4.2.21}$$

where $\widehat{F}^a_{\mu\nu}$, $\widehat{F}^b_{\mu a}$ and $\widehat{F}^c_{ab}$ are elements of $C^\infty(\mathcal{U})$. The decomposition (4.2.17) is very nicely adapted to the local expression of $\widehat{F}$ in the mixed local basis. Indeed, one shows that each term of this decomposition is directly associated to a component of $\widehat{F}_{\text{loc}}$ written in the basis $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\text{loc}})$. Thus, one obtains

$$\widehat{F}^a_{\mu\nu} = (\rho^* \widehat{R})^a_{\mu\nu} = \frac{1}{2}\left(\partial_\mu A^a_\nu - \partial_\mu A^a_\nu + A^b_\mu A^c_\nu C^a_{bc} - \tau^a_b(\partial_\mu \mathring{A}^b_\nu - \partial_\mu \mathring{A}^b_\nu + \mathring{A}^d_\mu \mathring{A}^e_\nu C^b_{de}\right) \tag{4.2.22}$$

$$\widehat{F}^b_{\mu a} = -((\rho^* \mathcal{D}\tau) \circ \mathring{\omega})^b_{\mu a} = -\left(\partial_\mu \tau^b_a + A^c_\mu \tau^d_a C^b_{cd} - \mathring{A}^c_\mu \tau^b_d\right) C^d_{ca} \tag{4.2.23}$$

$$\widehat{F}^c_{ab} = (\mathring{\omega}^* R_\tau)^c_{ab} = \frac{1}{2}\left(\tau^c_d C^c_{ab} - \tau^d_a \tau^e_b C^c_{de}\right) \tag{4.2.24}$$

Note that the first term has only Greek indices, the second term is a mix term since it contains both a Greek and a Latin index and the third term does contain only Latin indices so that it defines a purely algebraic component.

Chapter 5

Scalar product for differential forms

This chapter is devoted to the construction of additional structures on Lie algebroids A and on its associated differential complexes which permit, *in fine*, to define of a scalar product on the space of differential forms. Such constructions are mainly related to the metric, the Hodge star operator and the integral. These structures are extensions of those already defined in the context of differential geometry where they play an essential role in the construction of gauge field theories. In the context of Lie algebroids, the construction of gauge field theories is also related to these structures. Then, they have to be generalized in order to take into account both the geometric and the algebraic richness of the Lie algebroids.

Gauge theories are concerned with the dynamics of fields defined on $\mathcal{M}$. We have already seen in section 3.3 that the local trivialization of a differential q-form ω gives local components $\omega_{\mu_1\dots\mu_r a_1\dots a_s} \in C^\infty(\mathcal{U})$, with $r+s=q$. Moreover, these components depend *a priori* on a choice of local trivialization map S. One of the goal of this chapter is to exhibit a systematic method which permits to associate to any differential forms defined on a Lie algebroid a dynamical field which is both globally defined on $\mathcal{M}$ and independent of any choice of trivialization.

There is a direct method to obtain a globally defined field out of a differential form on A. It is given by the *inner integral* operator. This operator makes an integration along the inner degrees of differential forms and then, it isolates the so-called *maximal inner* component of ω. One shows that this maximal inner component is globally defined on $\mathcal{M}$ and is independent of any choice of trivialization. If the inner integral is composed with an integral over $\mathcal{M}$, one thus obtains an integration over A. This integration helps to define a gauge invariant action functional . Unfortunately, the inner integral operator gives non-trivial results only if the degree of form of ω is greater than the dimension of the fiber $\mathcal{L}$. If not, it trivially gives zero. Then, this integral should not be used to construct a gauge theory based on connections, covariant derivatives and curvatures on transitive Lie algebroids. To do so, additional structures are required.

In differential geometry, the Hodge star operator establishes an isomorphism of vector spaces between the space $\Omega^q(\mathcal{M})$ of differential forms defined on $\Gamma(T\mathcal{M})$ with values in $C^\infty(\mathcal{M})$ and the space $\Omega^{m-q}(\mathcal{M})$, where m denotes the dimension of $\mathcal{M}$. On transitive Lie algebroids, the corresponding generalization of the Hodge star operator takes into account both the geometry of $\mathcal{M}$ and the geometry of $\mathcal{L}$. It results in an operator which realizes an isomorphism of vector spaces between $\Omega^q(\mathsf{A})$ and $\Omega^{m+n-q}(\mathsf{A})$, easily extended to a map $\Omega^q(\mathsf{A},\mathsf{L}) \to \Omega^{m+n-q}(\mathsf{A},\mathsf{L})$. The action of the Hodge star operator consists in mapping the coefficients associated to the local description of differential forms, from their respective degrees of forms q to the corresponding space of degree $(m-q)$, up to combinatorial coefficients. In order to preserve the global structure of the differential complex on A, a metric $\widehat{g}$ on A is necessary.

On the base manifold $\mathcal{M}$ and on the kernel L, we define a geometric metric $g : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$ and an inner metric $h : \mathsf{L} \times \mathsf{L} \to C^\infty(\mathcal{M})$, respectively. In the general case, it is not possible to globally define a metric $\widehat{g}$ on a transitive Lie algebroid A as the "direct sum" of a metric g over $\mathcal{M}$ and a metric h on L. However, we show that any inner non-degenerate metric on A is related to a connection $\mathring{\nabla} : \Gamma(T\mathcal{M}) \to \mathsf{A}$, called the *metric connection on* A, so that the metric $\widehat{g}$ can be written as $\widehat{g} = \rho^* g + \mathring{\omega}^* h$, where $\mathring{\omega}$ denotes the ordinary connection 1-form associated to $\mathring{\nabla}$.

This chapter is constructed as follows. Firstly, we define metrics on A and we see how inner non-degenerate metrics are related to the existence of a connection $\mathring{\nabla}$ on A. This metric connection associated to $\widehat{g}$ will play the role of the "background" connection, the one used to define the so-called mixed local basis on $\mathsf{TLA}(\mathcal{U}, \mathfrak{g})$. If A is inner-orientable, this mixed local basis is also used to construct a globally-defined volume form on A. Secondly, according to this volume form, we define the inner integral operator acting on $\Omega^\bullet(\mathsf{A})$ and on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$, which will permit to define an integration over A. Thirdly, using an inner non-degenerate metric $\widehat{g}$, we define the Hodge star operator acting on $\Omega^\bullet(\mathsf{A})$ and on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. This definition is given locally, with respect to an atlas of Lie algebroids. To finish, we show that the composition of a Hodge star operator with an integral operator over A gives a scalar product defined on any differential complexes. Accordingly to this scalar product, one shows that forms of distinct degrees are orthogonals.

5.1 Metric on transitive Lie algebroids

In this section, we give the definition of metrics and inner-metrics on A. These metrics allow to define the orientation and the inner-orientation of A. An important result of this section establishes that any inner non-degenerate metric $\widehat{g}$ defined on A can be written as the direct sum of a metric on $\mathcal{M}$ and a metric on L. This expression is obtained by using the so-called metric connection on A associated to $\widehat{g}$.

5.1.1 Metrics and inner-metrics defined on A

As a vector bundle over $\mathcal{M}$, the Lie algebroid $\mathcal{A}$ can be equipped with a metric $\widehat{g}$ which is a $C^\infty(\mathcal{M})$-linear map which acts on the space of sections A as $\widehat{g} : \mathsf{A} \times \mathsf{A} \to C^\infty(\mathcal{M})$. *A priori*, this metric is locally defined *i.e.* it depends of the point $p \in \mathcal{M}$. This metric is said to be non-degenerate on A if $\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = 0$ for any $\mathfrak{Y} \in \mathsf{A}$, if and only if $\mathfrak{X} = 0$. A metric defined on a totally intransitive Lie algebroid L is said to be an *inner metric* on L.

A metric $\widehat{g}$ defined on A automatically defines a metric on its kernel L. This induced metric is called the *inner-metric associated to* $\widehat{g}$ and it is constructed with the pull-back by the injective map $\iota : \mathsf{L} \to \mathsf{A}$ as $h = \iota^* \widehat{g} : \mathsf{L} \times \mathsf{L} \to C^\infty(\mathcal{M})$ so that, for any $\gamma, \eta \in \mathsf{L}$, $h(\gamma, \eta) = \widehat{g}(\iota(\gamma), \iota(\eta))$. The non-degeneracy of h on L is defined as for $\widehat{g}$. A metric $\widehat{g}$ on A is said to be *inner non-degenerate* if the inner metric $h = \iota^* \widehat{g}$ associated to $\widehat{g}$ is non-degenerate on L.

A metric on $\mathcal{M}$, usually denoted as $g : \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to C^\infty(\mathcal{M})$, also defines a metric on A given by the pull-back by the anchor $\rho : \mathsf{A} \to \Gamma(T\mathcal{M})$ as $\widehat{g} = \rho^* g : \mathsf{A} \times \mathsf{A} \to C^\infty(\mathcal{M})$ defined, for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, as $\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y}))$. This induced metric $\widehat{g}$ is degenerate since it gives zero for $\mathfrak{X}$ or $\mathfrak{Y}$ are elements of L. In this sense, one says that the metric $\widehat{g}$ does not "see" the inner part of A.

Let $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$ and let h be an inner metric on L. For any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, the metric $\widehat{g} = \omega^* h$ is defined on A as $\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = h(\omega(\mathfrak{X}), \omega(\mathfrak{Y}))$. Here again, this metric is degenerate due to $\ker(\omega)$. [1] If ω is an ordinary connection 1-form on A normalized on L, then one shows that $h = \iota^* \widehat{g}$ so that h is the inner metric on L associated to $\widehat{g}$. In this case, the metric $\widehat{g}$ does not "see" the horizontal subspace of A associated to this connection.

An inner metric h defined on L can be extended on the differential space $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ of forms defined on A with values in L. It results in a graded $C^\infty(\mathcal{M})$-linear application $h : \Omega^\bullet(\mathsf{A}, \mathsf{L}) \times \Omega^\bullet(\mathsf{A}, \mathsf{L}) \to \Omega^\bullet(\mathsf{A})$. For any $\omega \in \Omega^p(\mathsf{A}, \mathsf{L})$ and $\eta \in \Omega^q(\mathsf{A}, \mathsf{L})$, the $(p+q)$-form $h(\omega, \eta) \in \Omega^{p+q}(\mathsf{A})$ is defined as

$$h(\omega, \eta)(\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}) = \frac{1}{p!q!} \epsilon^{a^1 a^2 \dots a^{p+q}} h\left(\omega(\mathfrak{X}_{a^1}, \dots, \mathfrak{X}_{a^p}), \eta(\mathfrak{X}_{a^{p+1}}, \dots, \mathfrak{X}_{a^{p+q}})\right) \quad (5.1.1)$$

for any $\mathfrak{X}_1, \dots, \mathfrak{X}_{p+q}$ where $\epsilon^{a^1 a^2 \dots a^{p+q}}$ denotes the Levi-Civita antisymmetric tensor with $\epsilon^{12 \dots p+q} = +1$.

A *Killing inner metric* on L is a metric which satisfies, for any $\gamma, \eta, \sigma \in \mathsf{L}$, the relation

$$h([\gamma, \eta], \sigma) + h(\eta, [\gamma, \sigma]) = 0. \quad (5.1.2)$$

This compatibility relation with the Lie bracket is an essential property for the construction of gauge-invariant theories. This Killing inner metric h is extended to the differential complex $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. It results in a map $h : \Omega^\bullet(\mathsf{A}, \mathsf{L}) \times \Omega^\bullet(\mathsf{A}, \mathsf{L}) \to \Omega^\bullet(\mathsf{A})$ whose compatibility with the graded Lie bracket on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ is given as

$$h([\omega_1, \omega_2], \omega_3) + (-1)^{k_1 k_2} h(\omega_2, [\omega_1, \omega_3]) = 0 \quad (5.1.3)$$

where $\omega_1 \in \Omega^{k_1}(\mathsf{A}, \mathsf{L})$, $\omega_2 \in \Omega^{k_2}(\mathsf{A}, \mathsf{L})$ and $\omega_3 \in \Omega^\bullet(\mathsf{A}, \mathsf{L})$.

5.1.2 Metric on a representation space

Consider a vector bundle $\mathcal{E} \to \mathcal{M}$ as a representation space for A, equipped with the representation $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$. As a vector bundle, we define on $\mathcal{E}$ a metric $h_\mathcal{E} : \Gamma(\mathcal{E}) \times \Gamma(\mathcal{E}) \to C^\infty(\mathcal{M})$. Similarly to the Killing metric, one can establish a compatibility relation between the representation map ϕ and the metric $h_\mathcal{E}$. A compatibility with the representation of all A would be too restrictive and, instead, we consider only a compatibility relation between $h_\mathcal{E}$ and the representation of the kernel L. Then, we say that a metric $h_\mathcal{E}$ is ϕ_L-*compatible* if

$$h_\mathcal{E}(\phi_\mathsf{L}(\ell) s_1, s_2) + h_\mathcal{E}(s_1, \phi_\mathsf{L}(\ell) s_2) = 0, \quad (5.1.4)$$

for any $s_1, s_2 \in \Gamma(\mathcal{E})$ and $\ell \in \mathsf{L}$. If one takes $\mathcal{E} = \mathcal{L}$ and ϕ the Lie bracket on L, one obtains exactly a Killing inner metric on L.

This metric can also be extended to forms defined on A with values in $\mathcal{E}$. This extended metric is defined as in (5.1.1). This metric will be used in chapter 7, to define the Lagrangian sector associated to (spinless) matter fields.

5.1.3 Inner orientation of a transitive Lie algebroid

The data of an inner metric h associated to a metric $\widehat{g}$ on A permits to define the inner orientation of A. This definition is related to both the local expression of h and to the gluing functions associated to an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i \in I}$.

With respect to this atlas of Lie algebroids, a metric h on L is locally trivialized over the open set $\mathcal{U}_i$ as $h_{\mathrm{loc}, i}(\gamma_i, \eta_i) = h(\Psi_i(\gamma_i), \Psi_i(\eta_i))$, where $\Psi_i : \Gamma(\mathcal{U}_i \times \mathfrak{g}) \to \mathsf{L}|_{\mathcal{U}_i}$ is

[1] As an application from $\mathsf{A} \to \mathsf{L}$, the kernel of ω is necessarily different from 0.

defined as in section 2.2.1. One denotes by $(E_1, E_2, \dots, E_n)$ a basis of the Lie algebra $\mathfrak{g}$ so that we define local components of the inner metric h as $(h_{\mathrm{loc},\,i})_{ab} = h_{\mathrm{loc},\,i}(E_a, E_b)$. These components are dynamical fields defined on $\mathcal{U}_i$. To simplify the notation, we simply denote the components $(h_{\mathrm{loc},\,i})_{ab}$ by $(h_i)_{ab} \in C^\infty(\mathcal{U}_i)$.

A *locally constant inner metric* is a metric h_{loc} defined on $\Gamma(\mathcal{U}_i \times \mathfrak{g})$ whose components $(h_i)_{ab} : \mathcal{U}_i \to \mathbb{R}$ are constant functions on $\mathcal{U}_i$. With respect to a local chart $(\mathcal{U}_i, \varphi_i)$ on $\mathcal{M}$, these components should fulfill the relation $\partial_\mu h_{ab} = 0$ for any $\mu = 1, \dots, m$ and $a, b = 1, \dots, n$, where $\partial_\mu = \partial/\partial x^\mu$ denotes an element of the basis of $T\mathcal{U}_i$.

Over the open set $\mathcal{U}_{ij}$, the inner metric h can be locally written either as $h_{\mathrm{loc},\,i}$, with respect to $(\mathcal{U}_i, \Psi_i)$, or as $h_{\mathrm{loc},\,j}$, with respect to $(\mathcal{U}_j, \Psi_j)$, respectively. These two trivializations describe the same object on L so that we have the relation

$$h_{\mathrm{loc},\,i}(\gamma_i, \eta_i) = h_{\mathrm{loc},\,j}(\gamma_j, \eta_j), \tag{5.1.5}$$

for any $\gamma_i, \eta_i \in \Gamma(\mathcal{U}_i \times \mathfrak{g})$ and $\gamma_j, \eta_j \in \Gamma(\mathcal{U}_j \times \mathfrak{g})$. Then,we show that the local components $(h_i)_{ab}$ and $(h_j)_{ab}$ of h are related by the formula:

$$(h_i)_{ab} = (G_{ji})^c_a (G_{ji})^d_b (h_j)_{cd}, \tag{5.1.6}$$

where the elements $(G_{ji})^b_a$ are the coefficients of the matrix-valued representation of the gluing function $\alpha^j_i : \Gamma(\mathcal{U}_{ij} \times \mathfrak{g}) \to \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$. These coefficients are elements of $C^\infty(\mathcal{U}_{ij})$.

As a matrix-valued element, the determinant of G_{ij} is well-defined on any open set $\mathcal{U}_{ij}$ and depends on the point $p \in \mathcal{U}_{ij}$. Then, a totally intransitive Lie algebroid L is said to be *orientable* if there exist an atlas $(\mathcal{U}_i, \Psi_i)_{i \in I}$ of L such that $\det(G_{ij}) > 0$ on any open sets $\mathcal{U}_{ij}$. A transitive Lie algebroid $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ is said to be *inner orientable* if its kernel L is orientable. A transitive Lie algebroid A is said to be *orientable* if it is inner orientable *and* if the base manifold $\mathcal{M}$ is orientable in the sense of the usual differential geometry.

5.1.4 Metric connection

In this section, we show that any inner-non degenerate metric $\widehat{g}$ on A is related to a unique connection on A called the *metric connection*. As we have seen in section 4.1.1, a connection on A is used to define a horizontal subspace which is the complement of L in A. Here, a *metric connection* associated to $\widehat{g}$ is used to define the *unique* horizontal subspace of A which is also *orthogonal* to L with respect to $\widehat{g}$.

Let $\widehat{g}$ be an inner non-degenerate metric on A and $h = \iota^*\widehat{g}$ be the inner metric associated to $\widehat{g}$. For any $\widetilde{\eta} \in \mathsf{L}$, consider the $C^\infty(\mathcal{M})$-linear map $f : \mathsf{L} \to C^\infty(\mathcal{M})$ defined as $f(\gamma) = h(\gamma, \widetilde{\eta})$, for any $\gamma \in \mathsf{L}$. For f and f' two maps corresponding to $\widetilde{\eta}$ and $\widetilde{\eta}'$, respectively, the non-degeneracy of h implies that they are equals if and only if $\widetilde{\eta} = \widetilde{\eta}'$. Then, the correspondence between the map f and the element $\widetilde{\eta}$ is unique. In [FLM13], one uses a variant of the theorem of Riesz to prove that any $C^\infty(\mathcal{M})$-linear map $f : \mathsf{L} \to C^\infty(\mathcal{M})$ can be written under the form $f(\gamma) = h(\gamma, \widetilde{\eta})$.

For any $\mathfrak{X} \in \mathsf{A}$, we define the $C^\infty(\mathcal{M})$-linear map $f(\gamma) = -\widehat{g}(\mathfrak{X}, \iota(\gamma))$ and we denote by $\mathring{\omega}(\mathfrak{X})$ the unique element of L associated to f by the previous construction. Then, one obtains

$$-\widehat{g}(\mathfrak{X}, \iota(\gamma)) = h(\mathring{\omega}(\mathfrak{X}), \gamma). \tag{5.1.7}$$

Since h is a non degenerate metric on L, $\mathring{\omega}$ is such that $\mathring{\omega} \circ \iota(\ell) = -\ell$, for any $\ell \in \mathsf{L}$ and thus, $\mathring{\omega}$ is normalized on L and correspond to an ordinary connection 1-form on A.

One denotes by $\mathring{\nabla} : \Gamma(T\mathcal{M}) \to \mathsf{A}$ the connection associated to $\mathring{\omega}$ and we call it *metric connection* on A.

The defining relation of $\mathring{\omega}$ can also be written as $\widehat{g}(\mathfrak{X} + \iota \circ \mathring{\omega}(\mathfrak{X}), \iota(\gamma)) = 0$. The first argument of $\widehat{g}$ is exactly the covariant connection $\mathring{\Theta}$ associated to $\mathring{\nabla}$ so that one has the following statement: for any $X \in \Gamma(T\mathcal{M})$ and any $\gamma \in \mathsf{L}$, we have

$$\widehat{g}(\mathring{\nabla}_X, \iota(\gamma)) = 0. \tag{5.1.8}$$

The connection $\mathring{\nabla}$ identifies the subspace of A which is both a complement space of L in A and also its orthogonal space, with respect to the metric $\widehat{g}$. This result relied on the single assumption that the metric $\widehat{g}$ was an inner non-degenerate metric. Hence, any inner-non degenerate metric $\widehat{g}$ on A is equivalent with the data of the triple $(g, h, \mathring{\nabla})$. In [Ker68], a similar metric is used to construct a non-abelian Kaluza-Klein theory on $\mathcal{P}$.

5.1.5 Decomposition of the inner non degenerate metric

The use of a metric connection on A associated to $\widehat{g}$ leads to the "sum" of a (possibly degenerate) metric g defined on the geometric component of A and a non-degenerate metric h defined on L. Here, we show that the decomposition of $\widehat{g}$ as $\rho^* g + \mathring{\omega}^* h$ is established by a direct computation.

Let $\widehat{g}$ be an inner non-degenerate metric on A and $\mathring{\nabla}$ the metric connection associated to $\widehat{g}$. Using the covariant connection $\mathring{\Theta}$ associated to $\mathring{\nabla}$, the metric $\widehat{g}$ can be decomposed, for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, as

$$\begin{aligned} \widehat{g}(\mathfrak{X}, \mathfrak{Y}) &= \widehat{g}(\mathring{\Theta}(\mathfrak{X}), \mathring{\Theta}(\mathfrak{Y})) + \widehat{g}(\mathring{\Theta}(\mathfrak{X}), \iota \circ \mathring{\omega}(\mathfrak{Y})) + \widehat{g}(\iota \circ \mathring{\omega}(\mathfrak{X}), \mathring{\Theta}(\mathfrak{Y})) + \widehat{g}(\iota \circ \mathring{\omega}(\mathfrak{X}), \iota \circ \mathring{\omega}(\mathfrak{Y})) \\ &= \widehat{g}(\mathring{\nabla}_{\rho(\mathfrak{X})}, \mathring{\nabla}_{\rho(\mathfrak{Y})}) + \widehat{g}(\iota \circ \mathring{\omega}(\mathfrak{X}), \iota \circ \mathring{\omega}(\mathfrak{Y})) \end{aligned}$$

Here, we have used the relations $\mathring{\Theta} = \rho^* \mathring{\nabla}$ and $\widehat{g}(\mathring{\Theta}(\mathfrak{X}), \iota \circ \ell) = 0$, for any $\mathfrak{X} \in \mathsf{A}$ and $\ell \in \mathsf{L}$. We define a metric g on $\mathcal{M}$ associated to the connection $\mathring{\nabla}$ as $g(X, Y) = \widehat{g}(\mathring{\nabla}_X, \mathring{\nabla}_Y)$, for any $X, Y \in \Gamma(T\mathcal{M})$. One writes the inner metric h associated to $\widehat{g}$ as $h = \iota^* \widehat{g}$ so that, for any $\mathfrak{X}, \mathfrak{Y} \in \mathsf{A}$, we obtain the following decomposition of the inner non degenerate metric as

$$\widehat{g}(\mathfrak{X}, \mathfrak{Y}) = g(\rho(\mathfrak{X}), \rho(\mathfrak{Y})) + h(\mathring{\omega}(\mathfrak{X}), \mathring{\omega}(\mathfrak{Y})). \tag{5.1.9}$$

Under this form, one says that the metric $\widehat{g}$ is block-diagonalized with respect to the geometric and the algebraic parts of A. The geometric metric g on $\mathcal{M}$ is possibly degenerate. To obtain the non-degeneracy of this metric, the metric $\widehat{g}$ on A has to be both inner non-degenerate *and* non-degenerate.

With respect to an atlas of Lie algebroids, this metric $\widehat{g}$ is locally trivialized over the open set $\mathcal{U}_i$ as $\widehat{g}_{\mathrm{loc},\,i} = S_i^* \widehat{g} : \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \times \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g}) \to C^\infty(\mathcal{U}_i)$. For any $X \oplus \gamma, Y \oplus \eta \in \mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$, the local trivialization $\widehat{g}_{\mathrm{loc}}$ is defined in terms of the metrics g_i on $\Gamma(\mathcal{U}_i)$ and $h_{\mathrm{loc},\,i}$ on $\Gamma(\mathcal{U}_i \times \mathfrak{g})$ as

$$\widehat{g}_{\mathrm{loc}}(X \oplus \gamma, Y \oplus \eta) = g_i(X, Y) + h_{\mathrm{loc},\,i}(\mathring{\omega}_{\mathrm{loc}}(X \oplus \gamma), \mathring{\omega}_{\mathrm{loc}}(Y \oplus \eta)) \tag{5.1.10}$$

With respect to a local chart, these two metrics are decomposed on the tensorial product between the basis $(\mathrm{d}x^1, \dots, \mathrm{d}x^m)$, associated to the cotangent bundle $T^*\mathcal{U}$ and the basis $(\theta^1, \dots, \theta^n)$, associated to the dual Lie algebra $\mathfrak{g}^*$. One would rather use the decomposition of $\widehat{g}_{\mathrm{loc}}$ on the mixed local basis associated to the metric connection $\mathring{\omega}_{\mathrm{loc}}$. Then, we obtain

$$\widehat{g}_{\mathrm{loc}} = g_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + h_{ab}\, \mathring{\omega}^a_{\mathrm{loc}} \otimes \mathring{\omega}^b_{\mathrm{loc}} \tag{5.1.11}$$

where $g_{\mu\nu} = g(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu})$ and h_{ab} are the local components of g and h_{loc}, respectively. With respect to a change of trivializations on $\mathcal{U}_{ij}$, these local expression of the metric g remain invariant and the local expressions of the metric h transforms as

$$h_{i,\,ab}\,\omega^a_{\text{loc},\,i} \otimes \omega^b_{\text{loc},\,i} = h_{j,\,ab}\,\omega^a_{\text{loc},\,j} \otimes \omega^b_{\text{loc},\,j} \tag{5.1.12}$$

Then, the local trivializations of the two metrics $g = \mathring{\nabla}^*\widehat{g}$ and $\mathring{\omega}^* h$ are correctly glued together to form globally-defined objects on $\mathcal{M}$.

5.2 Integration over A

5.2.1 The mixed local basis

In the previous examples, the mixed local basis associated to a background connection $\mathring{\omega}$ was used to give a convenient local description of globally-defined differential forms defined on A. Actually, this basis is an essential feature for the local trivializations of non-trivial transitive Lie algebroids.

As we've seen before, differential complexes on A are locally decomposed with respect to an atlas of Lie algebroids and a local chart of $\mathcal{M}$ on the graded tensorial product of a basis associated to the cotangent bundle and a basis associated to the dual of $\mathfrak{g}$ as in the formula (3.3.9). Contrary to the basis of $T^*\mathcal{U}$, the gluing relations associated to an element $\theta^a : \mathsf{TLA}(\mathcal{U} \times \mathfrak{g}) \to C^\infty(\mathcal{M})$, does not preserve this bi-graduation. Indeed, by changes of trivialization, the map θ^a moves to $\alpha^a_b \circ \theta^b + \chi^a_\mu \mathrm{d}x^\mu$, where $\chi^a_\mu \mathrm{d}x^\mu$ is equipped with a geometric degrees of forms. Then, the gluing transformations of θ^a do not preserve the dual space $\mathfrak{g}^*$.

Trivialization	S_1		S_2
Geometric degree of forms	$\mathrm{d}x^\mu$	$\longrightarrow$	$\mathrm{d}x^\mu$
Algebraic degree of forms	θ^a	$\longrightarrow$	$\alpha^a_b\,\theta^b + \chi^a_\mu \mathrm{d}x^\mu$

Thus, written in the basis $(\mathrm{d}x^\mu, \theta^a)$, the local components of ω support very badly changes of local trivialization.

This section gives the general theory of the mixed local basis applied to forms defined on A with values in $C^\infty(\mathcal{M})$. This theory can easily be extended to differential forms with values in L. As an application of this construction, we define a *volume form* on A and the so-called maximal inner differential form.

Let A be a transitive Lie algebroid equipped with an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I}$. Locally, we write the local expression a q-form $\omega \in \Omega^q(\mathsf{A})$ over the open set $\mathcal{U}$ as:

$$\omega_{\text{loc}} = \sum_{r+s=q} (\omega_{\text{loc}})^{\theta\,(r,s)}_{\mu_1\mu_2\dots\mu_r a_1 a_2\dots a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \ldots \wedge \mathrm{d}x^{\mu_r} \otimes \theta^{a_1} \wedge \theta^{a_2} \wedge \ldots \wedge \theta^{a_s} \tag{5.2.1}$$

where $(\mathrm{d}x^1, \mathrm{d}x^2, \ldots, \mathrm{d}x^m)$ and $(\theta^1, \theta^2, \ldots, \theta^n)$ are the basis of the cotangent bundle $T^*\mathcal{U}$ and the dual Lie algebra $\mathfrak{g}^*$, respectively, and the elements $(\omega_{\text{loc}})^{\theta\,(r,s)}_{\mu_1\mu_2\dots\mu_r a_1 a_2\dots a_s} \in C^\infty(\mathcal{U}_i)$ are the local components of ω. Under this form, one says that ω is locally trivialized on the basis $(\mathrm{d}x^\mu, \theta^a)$.

The background connection used to construct the mixed local basis is an ordinary connection 1-form $\mathring{\omega} \in \Omega^1(\mathsf{A}, \mathsf{L})$, which can be locally written over $\mathcal{U}_i$ as $\mathring{\omega}_{\text{loc},\,i} = (\mathring{A}^a_i -$

$\theta_i^a) \otimes E_a$, where $\mathring{A}_i^a \in \Omega^1(\mathcal{U}_i)$, $\theta_i^a \in \mathfrak{g}^*$ and $(E_a)_{a=1,\dots,n}$ denotes a basis for the Lie algebra $\mathfrak{g}$. In this notation, one has $\mathring{\omega}_{\mathrm{loc},\,i} = \mathring{\omega}^a_{\mathrm{loc},\,i} \otimes E_a$ where $\mathring{\omega}^a_{\mathrm{loc},\,i} = \mathring{A}_i^a - \theta_i^a$ is an element of $\Omega^1_{\mathsf{TLA}}(\mathcal{U}_i)$. A family of such 1-forms $(\mathring{\omega}^a_{\mathrm{loc},\,i})_{i\in I}$ is called the *mixed local basis* relative to the connection $\mathring{\nabla}$ and to the basis $(E_a)_{a=1,\dots,n}$. Note that the set $(\mathring{\omega}^a_{\mathrm{loc},\,i})_{i\in I}$ *must* be given by the local trivialization of a connection 1-form defined on A.

We write the element $\theta^a \in \mathfrak{g}^*$ as $\theta^a = \mathring{A}^a - \mathring{\omega}^a_{\mathrm{loc}}$. In order to obtain the local trivialization of ω in the basis $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\mathrm{loc}})$ or, introducing the terminology, in the *mixed local basis*, one simply substitutes θ^a by its corresponding expression $\mathring{A}^a - \mathring{\omega}^a_{\mathrm{loc}}$. It results in the expression

$$\omega_{\mathrm{loc}} = \sum_{r+s=q} (\omega_{\mathrm{loc}})^{(r,s)}_{\mu_1\mu_2\dots\mu_r a_1 a_2\dots a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_r} \wedge \mathring{\omega}^{a_1}_{\mathrm{loc}} \wedge \mathring{\omega}^{a_2}_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^{a_s}_{\mathrm{loc}} \quad (5.2.2)$$

where $(\omega_{\mathrm{loc}})^{(r,s)}_{\mu_1\mu_2\dots\mu_r a_1 a_2\dots a_s} \in C^\infty(\mathcal{M})$ are the *new* components of ω_{loc} with respect to this mixed local basis.

Over the open set $\mathcal{U}_{ij}$, local trivializations of the brackground connection 1-form $\mathring{\omega}$ are glued as $\mathring{\omega}_{\mathrm{loc},\,i} = \alpha^i_j \circ \mathring{\omega}_{\mathrm{loc},\,j} \circ s^j_i$. Using the representation of α^i_j as a matrix-valued function, one obtains $\mathring{\omega}^a_{\mathrm{loc},\,i} = (G_{ij})^a_b (\mathring{\omega}^b_{\mathrm{loc},\,j}) \circ s^j_i$. Then, the basis used in the decomposition of ω in the mixed local basis are preserved by changes of trivializations.

Trivialization	S_1		S_2
Geometric degree of forms	$\mathrm{d}x^\mu$	$\longrightarrow$	$\mathrm{d}x^\mu$
Algebraic degree of forms *given by the mixed local basis*	$\mathring{\omega}^a$	$\longrightarrow$	$\alpha^a_b\, \omega^b \circ s^2_1$

5.2.2 Volume form on A

A volume form on A essentially plays the same role as the volume form on $\mathcal{M}$: it permits an integration of differential forms on A. For the moment, since the integration of differential forms has not been defined yet, a volume form is defined as a differential form of degrees $n = \dim(\mathcal{L})$ on A which permits to canonically exhibit the so-called *maximal inner term* associated to any $\omega \in \Omega^q(\mathsf{A}, \mathsf{L})$.

Let A be an inner-orientable transitive Lie algebroid equipped with an inner metric h. We consider a background connection 1-form $\mathring{\omega}$ and one denotes by $\mathring{\omega}_{\mathrm{loc},\,i}$ its local trivialization. Let $\mathring{\omega}^1_{\mathrm{loc},\,i} \wedge \mathring{\omega}^2_{\mathrm{loc},\,i} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc},\,i}$ be a n-form defined on $\mathsf{TLA}(\mathcal{U}_i, \mathfrak{g})$, with values in $C^\infty(\mathcal{U}_i)$. On the open set $\mathcal{U}_{ij}$, the gluing relations of each $\mathring{\omega}^a_{\mathrm{loc},\,i}$ are given by the equation (3.3.7) so that the corresponding n-form defined in $\Omega^s_{\mathsf{TLA}}(\mathcal{U}_j)$ is of the form

$$\begin{aligned}
\mathring{\omega}^1_{\mathrm{loc},\,j} \wedge \mathring{\omega}^2_{\mathrm{loc},\,j} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc},\,j} &= (G_{ji})^1_{a_1} (G_{ji})^2_{a_2} \cdots (G_{ji})^n_{a_n} \left(\mathring{\omega}^{a_1}_{\mathrm{loc},\,i} \wedge \mathring{\omega}^{a_2}_{\mathrm{loc},\,i} \wedge \dots \wedge \mathring{\omega}^{a_n}_{\mathrm{loc},\,i} \right) \circ s^i_j \\
&= (G_{ji})^1_{a_1} (G_{ji})^2_{a_2} \cdots (G_{ji})^n_{a_n} \epsilon^{a_1 a_2 \dots a_n} \left(\mathring{\omega}^1_{\mathrm{loc},\,i} \wedge \mathring{\omega}^2_{\mathrm{loc},\,i} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc},\,i} \right) \circ s^i_j \\
&= \det(G_{ji}) \left(\mathring{\omega}^1_{\mathrm{loc},\,i} \wedge \mathring{\omega}^2_{\mathrm{loc},\,i} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc},\,i} \right) \circ s^i_j
\end{aligned}$$

One recalls that the local inner metrics $(h_i)_{ab}$ and $(h_j)_{ab}$ are related over the open set $\mathcal{U}_{ij}$ as $(h_i)_{ab} = (G_{ji})^c_a (G_{ji})^d_b (h_j)_{cd}$ so that we have

$$\det(h_{\mathrm{loc},\,i}) = \det(G_{ji})^2 \det(h_{\mathrm{loc},\,j}). \quad (5.2.3)$$

Moreover, as a consequence of the chain relations (2.2.7), one has $\det(G_{ij}) = \det(G_{ji})^{-1}$. Then, one defines the *volume form* ω_{vol} on A as the set of local 1-forms $((\omega_{\text{vol}})_i)_{i\in I}$ defined on the open sets $\mathcal{U}_i$ as

$$(\omega_{\text{vol}})_i = \sqrt{\det(h_{\text{loc},\,i})}\, \mathring{\omega}^1_{\text{loc},\,i} \wedge \mathring{\omega}^2_{\text{loc},\,i} \wedge \dots \wedge \mathring{\omega}^n_{\text{loc},\,i} \tag{5.2.4}$$

It is straightforward to prove that, on the open set $\mathcal{U}_{ij}$, the elements $(\omega_{\text{vol}})_i$ and $(\omega_{\text{vol}})_j$ are related as

$$\begin{aligned}(\omega_{\text{vol}})_j &= \sqrt{\det(h_{\text{loc},\,j})}\, \mathring{\omega}^1_{\text{loc},\,j} \wedge \mathring{\omega}^2_{\text{loc},\,j} \wedge \dots \wedge \mathring{\omega}^n_{\text{loc},\,j} \\ &= \sqrt{\det(h_{\text{loc},\,i})}\, |\det(G_{ij})| \det(G_{ji}) \left(\mathring{\omega}^1_{\text{loc},\,i} \wedge \mathring{\omega}^2_{\text{loc},\,i} \wedge \dots \wedge \mathring{\omega}^n_{\text{loc},\,i}\right) \circ s^i_j \\ &= (\omega_{\text{vol}})_i \circ s^i_j\end{aligned}$$

From section 3.3.1, this last result implies that the differential forms $((\omega_{\text{vol}})_i)_{i\in I}$ are the local trivializations of a global form $\omega_{\text{vol}} \in \Omega^n(\mathsf{A})$.

5.2.3 Maximal inner form on A

We consider a q-form $\omega \in \Omega^q(\mathsf{A})$ which is locally trivialized on $\mathcal{U}_i$ in the mixed local basis as in formula (5.2.2). Over each open set $\mathcal{U}_i$, we are interested in the component of $\omega_{\text{loc},\,i}$ associated to the bi-graduation $(q-n, n)$ *i.e.* which can be written as $\omega^{(q-n,n)}_{\text{loc},\,i} = (\omega_{\text{loc},\,i})^{(q-n,n)}_{\mu_1\dots\mu_{q-n}a_i\dots a_n} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_{q-n}} \otimes \mathring{\omega}^{a_1}_{\text{loc}} \wedge \dots \wedge \mathring{\omega}^{a_n}_{\text{loc}}$. This local component is written in terms of ω_{vol} as

$$\omega^{(q-n,n)}_{\text{loc},\,i} = \frac{n!}{\sqrt{\det(h_{\text{loc},\,i})}} (\omega_{\text{loc},\,i})^{(q-n,n)}_{\mu_1\mu_2\dots\mu_{q-n}} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_{q-n}} \wedge (\omega_{\text{vol}})_i \tag{5.2.5}$$

for each $i \in I$, where $(\omega_{\text{vol}})_i$ denotes the local restriction of ω_{vol} over $\mathcal{U}_i$. note that, on the local component of $\omega^{(q-n,n)}_{\text{loc},\,i}$, the Latin indices have been contracted to give the factor $n!$ so that they do not occur anymore.

We denote $\omega^{m.i.}_{\text{loc},\,i} = \frac{1}{\sqrt{\det(h_{\text{loc},\,i})}} (\omega_{\text{loc},\,i})^{(q-n,n)}_{\mu_1\mu_2\dots\mu_{q-n}}$. This corresponds to the *maximal inner component* associated to ω. Then, the $(q-n,n)$-component of ω can be written as

$$\omega^{q-n,n}_{\text{loc},\,i} = n!\, \omega^{m.i.}_{\text{loc},\,i} \wedge \omega_{\text{vol},\,i} \tag{5.2.6}$$

The notation $m.i.$ stands for "maximal inner".

By changes of trivialization, a straightforward computation shows that the maximal inner component $\omega^{\text{m.i.}}_i$ and $\omega^{m.i.}_j$ associated to ω are related by $\omega^{\text{m.i.}}_i = \omega^{\text{m.i.}}_j$. Then, the family of forms $(\omega^{\text{m.i.}}_i)_{i\in I}$ are the local trivializations of a global form $\omega^{\text{m.i.}} \in \Omega^{q-n}(\mathcal{M})$.

One shows that the existence of a maximal inner term is independent of the background connection. Indeed, a q-form ω can be locally described either with respect to the basis $(\mathrm{d}x^\mu, \theta^a)$ as

$$\omega_{\text{loc}} = (\omega_{\text{loc}})^{\theta,(q-n,n)}_{\mu_1\dots\mu_{q-n}} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_{q-n}} \otimes \theta^{a_1} \wedge \dots \wedge \theta^{a_n} + \dots \tag{5.2.7}$$

where "$\dots$" denotes the components of ω_{loc} with inner degrees which are *not* maximal, or, with respect to the basis $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\text{loc}})$, as

$$\omega_{\text{loc}} = n!\, \omega^{\text{m.i.}} \sqrt{\det(h_{\text{loc}})}\, \mathring{\omega}^1_{\text{loc}} \wedge \mathring{\omega}^2_{\text{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\text{loc}} + \dots \tag{5.2.8}$$

By writing $\theta^a = A^a - \mathring{\omega}^a$, it is straightforward to see that

$$\omega^{\text{m.i.}} = \frac{(-1)^n}{n!\sqrt{\det(h_{\text{loc}})}} \, (\omega_{\text{loc}})^{\theta,(q-n,n)}_{\mu_1\dots\mu_{q-n}} \mathrm{d}x^{\mu_1} \wedge \dots \wedge \mathrm{d}x^{\mu_{q-n}} \tag{5.2.9}$$

Then, the existence of $\omega^{\text{m.i.}}$ does not depend on the background connection $\mathring{\nabla}$ used to define the mixed local basis but only on the assumption that A is inner-orientable. Depending on the choice of basis (mixed local basis or not), the $(q-n)$-form defined on $\mathcal{M}$ can be directly "read" on the local decomposition of ω.

5.2.4 Inner integration

The *inner integration* defined on $\Omega^\bullet(\mathsf{A})$ is an integration in the sense where it gets rid of the inner degrees of form. It consists in extracting the maximal inner terms associated to any element $\omega \in \Omega^\bullet(\mathsf{A})$. Then, this inner operation is defined as

$$\int_{inner} : \Omega^\bullet(\mathsf{A}) \to \Omega^{\bullet-n}(\mathcal{M}) \quad ; \quad \omega \mapsto \omega^{\text{m.i.}} \tag{5.2.10}$$

where $\omega_i^{\text{m.i.}} \in \Omega^{q-n}(\mathcal{U}_i)$ denotes the maximal inner component of $\omega_{\text{loc},\,i}$. By construction, we remark that $\int_{inner} \omega_{\text{vol}} = 1$. In [Kub98], a similar operator is constructed by using a "dual" volume form $\epsilon \in \wedge^n \mathsf{L}$. This inner integral is naturally extended to $\Omega^\bullet(\mathsf{A},\mathsf{L})$ as

$$\int_{inner} : \Omega^\bullet(\mathsf{A},\mathsf{L}) \to \Omega^{\bullet-n}(\mathcal{M},\mathcal{L}) \quad ; \quad \omega \mapsto \omega^{\text{m.i.}} \tag{5.2.11}$$

This comes from the fact that, for any q-form ω, the term $\omega_i^{\text{m.i.}} \in \Omega^{q-n}(\mathcal{U}_i, \mathfrak{g})$ is related to $\omega_j^{\text{m.i.}}$ over $\mathcal{U}_{ij}$ as $\omega_i^{\text{m.i.}} = \alpha^i_j \omega_j^{\text{m.i.}}$. Then, the elements $(\omega_i^{\text{m.i.}})_{i\in I}$ are the local trivializations of defined on $\mathcal{M}$ with value in L.

By construction of this inner integration, a lot of informations related to the degrees of freedom of ω are lost after performing this integral. Indeed, every local components of ω which does not contain the local volume form $(\omega_{\text{vol}})_{\text{loc}}$ are "killed" by this inner integration. Then, any q-form ω defined on A with degree $q < n$ has no maximal inner component so that it is in the kernel of this inner integral.

Thus, it seems like that the kernel of this integral is "too big" to be related to the construction of gauge field theories. Indeed, these are usually related to objects with low degrees in differential form, for instance connections or curvatures which are algebraically described in terms of 1-forms or 2-forms on A. Then, the application of the inner integral on these objects will give zero as soon as the dimension of the inner degrees of freedom becomes greater than 2 or 3.

Nevertheless it results in the action of this inner integral an interesting feature for the construction of physical models: the resulting maximal inner form is globally defined on $\mathcal{M}$. Then, before using this operator to construct gauge theories, additional structures have to be defined. This will be done in section 5.3 by the introduction of the Hodge-star operator.

5.2.5 Integration over A

The integration over A permits to associate to any differential q-form defined on A a scalar number. However, by construction, this integral inherits the properties of the inner integral, in particular concerning the loss of informations related to ω. This will be solved in the definition of the scalar product in section 5.4.

Let A be an orientable transitive Lie algebroids. The integration of $\omega \in \Omega^q(\mathsf{A})$ over A is given by the composition of the inner integral operator with an integration over $\mathcal{M}$ as defined in 1.1.4 so that we have

$$\int_{\mathsf{A}} \omega = \int_{\mathcal{M}} \circ \int_{\text{inner}} \omega = \int_{\mathcal{M}} \omega^{\text{m.i.}} \in \mathbb{R} \tag{5.2.12}$$

Obviously, this definition makes sense only if the integral over $\mathcal{M}$ converges, which is the case when $\mathcal{M}$ is compact or if $\omega^{\text{m.i.}}$ has compact support.

The integration over $\mathcal{M}$ gives a non-zero result only for forms with maximal degrees of geometric forms. Then, the integration over A does not vanish only for q-forms ω with both maximal degrees in the inner directions *and* in the de Rham direction. This is only the case for $q = m + n$ where $m = \dim(\mathcal{M})$ and $n = \dim(\mathcal{L})$.

5.3 Hodge star operator

The definition of a Hodge star operator on a Lie algebroid extends the geometric definition of the Hodge star operator on $\mathcal{M}$. This extension takes into account both the geometric and the algebraic degrees of forms of differential complexes defined on A. It establishes an isomorphism of vector spaces between forms of degree q with forms of degree $m + n - q$. Roughly speaking, it consists in transferring each local components of a differential form from a bi-graduation (r, s) to bi-graduation $(m-r, n-s)$, up to combinatorial coefficients.

In order to preserve the global structure on A, these components should be given in the mixed local basis associated to a metric connection on A.

Let A be an orientable transitive Lie algebroid equipped with a non degenerate and inner non-degenerate metric $\widehat{g}$ which is decomposed as $(g, h, \mathring{\nabla})$ and an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I}$. Both the metrics g and h are non-degenerate. The *Hodge star operator* $\star : \Omega^q(\mathsf{A}) \to \Omega^{q-m-n}(\mathsf{A})$ associated to the metric $\widehat{g}$ is a $C^\infty(\mathcal{M})$-linear map is defined as follows.

Let $\omega \in \Omega^q(\mathsf{A})$ be a q-form locally trivialized, in a given chart, with respect to the mixed local basis to $\omega_{\text{loc}} \in \Omega^q_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$ as:

$$\omega_{\text{loc}} = \sum_{r+s=q} (\omega_{\text{loc}})^{(r,s)}_{\mu_1\mu_2\dots\mu_r a_1 a_2 \dots a_s} \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \dots \wedge \mathrm{d}x^{\mu_r} \wedge \mathring{\omega}^{a_1}_{\text{loc}} \wedge \mathring{\omega}^{a_2}_{\text{loc}} \wedge \dots \wedge \mathring{\omega}^{a_s}_{\text{loc}} \tag{5.3.1}$$

The Hodge-star operator acts on each local components of ω_{loc} such that, the form $\star\,\omega_{\text{loc}} \in \Omega^{m+n-p}_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$ is defined as

$$\begin{aligned}\star\,\omega_{\text{loc}} = \sum_{r+s=p} (-1)^{s(m-r)} \frac{1}{r!s!} \sqrt{\det(h_{\text{loc}})}\sqrt{\det(g)}\, (\omega_{\text{loc}})^{(r,s)}_{\mu_1\dots\mu_r a_1 \dots a_s}\, \epsilon_{\nu_1\dots\nu_m}\, \epsilon_{b_1\dots b_n} \\ \times g^{\mu_1\nu_1} \cdots g^{\mu_r\nu_r}\, h^{a_1 b_1} \cdots h^{a_s b_s}\, \mathrm{d}x^{\nu_{r+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_m} \wedge \mathring{\omega}^{b_{s+1}}_{\text{loc}} \wedge \cdots \wedge \mathring{\omega}^{b_n}_{\text{loc}}\end{aligned} \tag{5.3.2}$$

where $\epsilon_{\nu_1\dots\nu_m}$ and $\epsilon_{b_1\dots b_n}$ are the Levi-Civita tensors and where $(g^{\mu\nu})$ and (h^{ab}) are the inverse matrix components of $(g_{\mu\nu})$ and (h_{ab}), respectively.

On the open set $\mathcal{U}_{ij}$, the differential form ω can be trivialized either over $\mathcal{U}_i$ with respect to the trivialization S_i, or over $\mathcal{U}_j$, with respect to the trivialization S_j. We apply the Hodge star operator on $\omega_{\text{loc},\,,i}$ and $\omega_{\text{loc},\,j}$ in order to compute $(\star\,\omega_{\text{loc}})_i$ and $(\star\,\omega_{\text{loc}})_j$. To this purpose, one uses the following gluing relations:

- $(h_j)_{ab} = (G_{ij})^c_a (G_{ij})^d_b (h_i)_{cd}$ and $(h_j)^{ab} = (G_{ji})^a_c (G_{ji})^b_d (h_i)^{cd}$
- $\sqrt{\det(h_{\text{loc},\, j})} = |\det(G_{ij})| \sqrt{\det(h_{\text{loc},\, i})}$
- $\mathring{\omega}^a_{\text{loc},\, j} = (G_{ji})^a_b (\mathring{\omega}^b_{\text{loc},\, i}) \circ s^i_j$
- $\omega_{j,\, \mu_1\mu_2\dots\mu_r a_1 a_2 \dots a_s} = (G_{ij})^{b_1}_{a_1} (G_{ij})^{b_2}_{a_2} \cdots (G_{ij})^{b_s}_{a_s} \alpha_{ji} \left(\omega_{i,\, \mu_1\mu_2\dots\mu_r b_1 b_2 \dots b_s}\right)$

By a direct computation, one obtains $(\star\omega_{\text{loc}})_j = (\star\omega_{\text{loc}})_i \circ s^i_j$ for any $i, j = 1, \dots, I$. Then, the forms $\{(\star\omega_{\text{loc}})_i\}_{i\in I}$ are the local trivializations of a global form $\star\omega \in \Omega^{q-n-m}(\mathsf{A})$ and the map $\star : \Omega^q(\mathsf{A}) \to \Omega^{q-m-n}(\mathsf{A})$ defined as $\omega \mapsto \star\omega$ is well-defined. By using cumbersome calculations, one shows that the map $\star$ is invertible. Indeed, for any $\omega \in \Omega^q(\mathsf{A})$ one has $\star\star\omega = (-1)^{(m+n-q)q}\omega$.

The Hodge star operator can be extended to $\Omega^q(\mathsf{A}, \mathsf{L})$ with merely the same definition. Thus, one obtains $\star : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^{q-m-n}(\mathsf{A}, \mathsf{L})$ defined as $\omega \mapsto \star\omega$ where $(\star\omega_{\text{loc}})_i \in \Omega^{q-m-n}_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$. It is invertible by the same manner as before.

5.4 Scalar product for forms defined on A

We arrive at the final result of this chapter: the definition of a scalar product for differential forms defined on A with values either in $C^\infty(\mathcal{M})$ or in L. This scalar product involve a non-degenerate and inner non-degenerate metric $\widehat{g}$ on A, a background connection used to define a mixed local basis and a Hodge star operator. With respect to this scalar product, differential forms of distinct degrees are orthogonal.

Let A be an orientable and inner-orientable transitive Lie algebroid equipped with a non-degenerate and inner-non degenerate metric $\widehat{g} = (g, h, \mathring{\nabla})$ and a Lie algebroid atlas $(\mathcal{U}_i, S_i)_{i\in I} = (\mathcal{U}_i, \Psi_i, \nabla^0_i)_{i\in I}$. For any $\omega \in \Omega^q(\mathsf{A})$ and $\eta \in \Omega^r(\mathsf{A})$, one defines the *scalar product* for differential forms with values in $C^\infty(\mathcal{M})$ as

$$\langle \omega, \eta \rangle = \int_{\mathsf{A}} (\omega \wedge \star\eta) \tag{5.4.1}$$

By construction, for $q = p$, the degrees of forms of ω are completed by the degrees of forms of $\star\eta$ so that $\omega \wedge \star\eta \in \Omega^{m+n}(\mathsf{A})$ and then, the integral over A does not necessarily vanish. However, for $q \neq r$, the scalar product gives 0. In this sense, differential forms with distinct degrees are orthogonal with respect to this scalar product. Using straightforward computations, one checks that this scalar product is symmetric.

This scalar product also gives rise to relations of orthogonality between the bi-graduations associated to the local decomposition of a differential form. To see this, we locally decompose a differential form ω in the mixed local basis $(\mathrm{d}x^\mu, \mathring{\omega}^a_{\text{loc}})$. Then, one shows that the local (r, s)-form, where r denotes the degree of forms on $\mathcal{U}$ and s denotes the degree of the mixed local basis $\mathring{\omega}^a_{\text{loc}}$, is orthogonal to any local form with bi-degrees (r', s') except for $r = r'$ and $s = s'$. In other words, the scalar product of two differential forms with the same degree of forms gives the product, terms to terms, of each components, up to combinatorial coefficients. In chapter 7, this result is directly applied to the definition of a gauge action functional.

By linearity, this scalar product is extended to differential complexes of forms defined on A. Let $\omega_1, \omega_2 \in \Omega^\bullet(\mathsf{A})$, as elements of a differential complex of forms, they can be written as $\omega_1 = \oplus_{a=0}\omega_1^a$ and $\omega_2 = \oplus_{a=0}\omega_2^a$, where ω_1^a and ω_2^a belong to $\Omega^a(\mathsf{A})$. Thus, the scalar product between these two differential complexes gives

$$\langle \omega_1, \omega_2 \rangle = \sum_{a=0} \langle \omega_1^a, \omega_2^a \rangle \tag{5.4.2}$$

By construction of the scalar product, no mix terms are involved.

This scalar product can be also extended to $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ with merely the same definition. Let $\omega \in \Omega^q(\mathsf{A}, \mathsf{L})$ and $\eta \in \Omega^r(\mathsf{A}, \mathsf{L})$, one has:

$$\langle \omega, \eta \rangle_h = \int_{\mathsf{A}} h(\omega, \star\eta) \in \mathbb{R} \tag{5.4.3}$$

Here, the metric h is defined on forms defined on A with values in L as in (5.1.1). Let $\omega_1 = \oplus_{a=0}\omega_1^a$ and $\omega_2 = \oplus_{a=0}\omega_2^a$ where ω_1^a and ω_2^a belong to $\Omega^a(\mathsf{A}, \mathsf{L})$. Then, one has:

$$\langle \omega_1, \omega_2 \rangle_h = \sum_{a=0} \langle \omega_1^a, \omega_2^a \rangle_h \tag{5.4.4}$$

This scalar product defined on differential forms with values in L will be a fundamental feature to construct gauge-invariant theory on transitive Lie algebroids.

Chapter 6

Gauge theory based on transitive Lie algebroids

In this chapter, we consider groups of symmetries which act on the *inner* space of physical systems. In the theory of fiber bundles, it is well-known that the symmetry is encoded in the structure group G associated to a principal bundle $\mathcal{P}$. The gauge action of this structure group is given by the group of vertical automorphisms $\{f : \mathcal{P} \to \mathcal{P} | f(u \cdot g) = f(u) \cdot g$ and $\pi(f(u)) = \pi(u),\ \forall u \in \mathcal{P}, \forall g \in G\}$ which act on objects defined on $\mathcal{P}$ such as the connections, the covariant derivatives, the curvatures, *etc.* Such an action is called the *geometric action* of the group G.

On transitive Lie algebroids, gauge groups do not always exist and, instead, we consider only *infinitesimal* gauge actions. These infinitesimal gauge actions are related to the kernel L as it represents the inner degrees of freedom of the Lie algebroids related to the infinitesimal "inner" displacements. These are given by the Cartan operation $(\mathsf{L}, i, \mathrm{L})$ and acts on the space of ordinary connections, covariant derivatives and curvatures by the Lie derivative. This corresponds to the *geometric infinitesimal action* of L. Restricted to the subspace of ordinary connections on A, it results in the usual gauge transformations of the differential geometry: connections transform with an inhomogeneous term, covariant derivatives preserve the gauge action of L on a representation space and curvatures transform, at the first order, in the adjoint action of L. On Atiyah Lie algebroids, theses transformations are exactly the infinitesimal version of the action of the gauge group $\mathcal{G}$ associated to $\mathcal{P}$.

On the space of generalized connections on A, the reduced kernel endomorphism associated to a generalized connection ϖ is not compatible, in a certain sense, with the geometric infinitesimal action of L. Even if the Lie derivative of L is well-defined on generalized connections, it induces "messy" gauge transformations on the various fields of the gauge theory. In particular, the induced ordinary connection associated to ϖ does not transform as an ordinary connection 1-form, due to the presence of the background connection $\mathring{\omega}$ which does no support action of L. Also, the generalized covariant derivative does not preserve the representation space of A. Finally, the transformations of the curvature associated to ϖ make improbable, if not impossible, the construction of a gauge theory based on it.

To solve these messy gauge transformations, we introduce a new gauge action of the kernel L. It does not correspond to a geometric action, and then, we call it the *algebraic infinitesimal action* of L. This algebraic gauge action is defined in order to preserve the generalized covariant derivative associated to ϖ. By induction, we compute the gauge transformations of generalized connections, their reduced kernel endomorphisms and their induced ordinary connection 1-form. With this algebraic action, the curvature $\widehat{F}$ associated to ϖ gets homogeneous gauge transformations. In particular, the three terms associated to the decomposition of $\widehat{F}$ (see section 4.2.3) transform independently as homogeneous terms. The geometric and the algebraic gauge actions of L coincide only for $\tau = 0$ *i.e.* restricted on the subspace of ordinary connections on A. In chapter 7, we use this algebraic action of L to construct gauge invariant quantities.

This chapter is organized as follows. We introduce the geometric infinitesimal action of L in terms of the action of a Lie derivative associated to the Cartan operation $(\mathsf{L}, i, \mathrm{L})$. This action is implemented both on ordinary connections and on generalized connections. On Atiyah Lie algrebroids, one shows that the geometric action of L corresponds to the infinitesimal action of the gauge group on $\mathcal{P}$. On generalized connections, the geometric action of L makes apparent some incompatibility relations which will be discussed in the body of the text. Then, the algebraic gauge action of L is defined on generalized connections. Its non-geometrical statue will also be discussed. In particular, one will see that for $\tau = 0$, geometric and algebraic gauge action of L coincide.

6.1 Geometric action of L on ordinary connections

In the context of transitive Lie algebroids, infinitesimal geometric action is naturally defined *via* the definition of a Lie derivative along L. Initially, the Lie derivative was a geometric operation acting on vector fields and covectors. Here, we consider the algebraic definition related to the Cartan operation $(\mathsf{L}, i, \mathrm{L})$ on $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$, as defined in section 3.2.4.

Let A be a transitive Lie algebroid over $\mathcal{M}$ with kernel L. Considering L as a totally intransitive Lie algebroid (with null anchor), one defines the Cartan operation $(\mathsf{L}, i, \mathrm{L})$ on A, where i denotes the inner operation and L denotes the Lie derivative. These two maps are defined as

$$i_\xi : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^{q-1}(\mathsf{A}, \mathsf{L}) \quad ; \quad i_\xi \omega(\mathfrak{X}_1, \dots, \mathfrak{X}_{q-1}) = \omega(\iota \circ \xi, \mathfrak{X}_1, \dots, \mathfrak{X}_{q-1})$$

$$\mathrm{L}_\xi = i_\xi \circ \widehat{\mathrm{d}} + \widehat{\mathrm{d}} \circ i_\xi : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^q(\mathsf{A}, \mathsf{L})$$

for any $\omega \in \Omega^q(\mathsf{A}, \mathsf{L})$, $\mathfrak{X}_1, \dots, \mathfrak{X}_{q-1} \in \mathsf{A}$ and $\xi \in \mathsf{L}$. Here, the differential operator $\widehat{\mathrm{d}} : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^{q+1}(\mathsf{A}, \mathsf{L})$ is associated to the ad-representation of A on L.

The *infinitesimal gauge action of* L on A is defined on any element $\mathfrak{X} \in \mathsf{A}$ as

$$\mathfrak{X} \mapsto \mathfrak{X}^\xi = \mathfrak{X} - \iota \circ \delta\mathfrak{X} \quad \text{where} \quad \delta\mathfrak{X} = [\xi, \mathfrak{X}], \tag{6.1.1}$$

with ξ denotes the *parameter of the infinitesimal gauge transformation.* Since the gauge transformations based on L are infinitesimal, the reader should keep in mind the formula $\xi^2 \simeq 0$. In the following, the status of *geometric action of* L is given when the infinitesimal gauge action of L is encoded in the action of the Lie derivative L.

The infinitesimal gauge action of L is given on the connection 1-forms ω by the geometric action of L as

$$\omega \mapsto \omega^\xi = \omega - \delta\omega \quad \text{where} \quad \delta\omega = \mathrm{L}_\xi \omega \tag{6.1.2}$$

with $\xi \in \mathsf{L}$ and $\mathrm{L}_\xi : \Omega^q(\mathsf{A}, \mathsf{L}) \to \Omega^g(\mathsf{A}, \mathsf{L})$. The first order term $\delta\omega$ can be written as $\delta\omega(\mathfrak{X}) = \mathrm{L}_\xi\omega(\mathfrak{X}) = [\xi, \omega(\mathfrak{X})] + [\xi, \mathfrak{X}]$ for any $\mathfrak{X} \in \mathsf{A}$ and $\xi \in \mathsf{L}$. Then, using intrinsic notations, we obtain the geometric transformations of the ordinary connection 1-forms:

$$\omega^\xi = \omega - [\xi, \omega] + \widehat{\mathrm{d}}\xi \tag{6.1.3}$$

The geometric action of L preserves the space of ordinary connection in the sense that $\omega^\xi(\iota \circ \ell) = -\ell$ for any $\xi, \ell \in \mathsf{L}$. Then ω^ξ is still normalized on L and still corresponds to an ordinary connection 1-form. However, it does not define the same horizontal subspace of A as the connection associated to ω.

The geometric gauge transformation of $\Theta : \mathsf{A} \to \mathsf{A}$, associated to $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$, is directly computed using the Lie derivative along $\xi \in \mathsf{L}$ as

$$\Theta \mapsto \Theta^{\xi} = \Theta - \delta\Theta \tag{6.1.4}$$

where $\delta\Theta = \mathrm{L}_{\xi}\Theta = [\xi, \Theta]$ and then, the covariant connection transforms as $\Theta^{\xi} = \Theta - [\xi, \Theta]$. Written under this form, one sees that Θ^{ξ} defines the horizontal subspace of A associated to the connection ω^{ξ} in the sense that $\omega^{\xi}(\Theta^{\xi}) = 0$, *at first order in* ξ. The geometric action of L on Θ is compatible with the geometric action on the ordinary connection 1-form ω in the sense that $\Theta^{\xi}(\mathfrak{X}) = \mathfrak{X} + \iota \circ \omega^{\xi}(\mathfrak{X})$, for any $\mathfrak{X} \in \mathsf{A}$ and $\xi \in \mathsf{L}$.

Let $\mathcal{E}$ be a representation space for A equipped with the representation ϕ on $\mathcal{E}$ so that the differential operator $\widehat{\mathrm{d}}_{\phi}$ is well-defined. The infinitesimal geometric gauge action of L on $s \in \Gamma(\mathcal{E})$ is given by the Lie derivative associated to $\widehat{\mathrm{d}}_{\phi}$ as

$$s^{\xi} = s - \delta s \quad \text{where} \quad \delta s = \mathrm{L}_{\xi} s = \widehat{\mathrm{d}}_{\phi} \circ i_{\xi} s + i_{\xi} \circ \widehat{\mathrm{d}}_{\phi} s = \phi_{\mathsf{L}}(\xi) s \tag{6.1.5}$$

and $\xi \in \mathsf{L}$ denotes the parameter of the gauge transformation. The infinitesimal geometric gauge transformation of the covariant derivative associated to the connection ∇ and the representation $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ is induced from the geometric action of L on the covariant connection Θ and on the element s as

$$\mathcal{D}^{\xi} s^{\xi} = \phi(\Theta^{\xi}) s^{\xi} \tag{6.1.6}$$

where $\xi \in \mathsf{L}$. Using a direct computation, we show that the geometric action of L on the covariant derivative is given in terms of ω as

$$\mathcal{D}^{\xi} s^{\xi}(\mathfrak{X}) = \phi(\mathfrak{X}) s^{\xi} - \phi_{\mathsf{L}}(\omega^{\xi}) s^{\xi}. \tag{6.1.7}$$

At the first order in ξ, the representation space of A is preserved by the geometric action of L in the sense that

$$\mathcal{D}^{\xi} s^{\xi} = \mathcal{D}s - \phi_{\mathsf{L}}(\xi)\mathcal{D}s + \mathcal{O}(\xi^2) = (\mathcal{D}s)^{\xi} + \mathcal{O}(\xi^2) \tag{6.1.8}$$

where $\mathcal{O}(\xi^2)$ denotes terms of higher order in ξ. This is a good thing since, initially, the covariant derivative was introduced in gauge field theories to obtain covariant objects with respect to the action of a local gauge group. In differential geometry, the geometric structure of the covariant derivative necessarily gives this covariance. In the context of Lie algebroids, this point is recovered for the geometric action on L on covariant derivative associated to ordinary connections on A.

We proceed to the computation of the geometric action of L on the curvature F associated to ω. As expected, we find the infinitesimal version of the well-known homogeneous gauge transformations of F. To see this, let $F = \rho^* R \in \Omega^2(\mathsf{A}, \mathsf{L})$ be the curvature associated to the connection ∇ on A. The infinitesimal gauge transformation on the L-horizontal 2-form F is given by the geometric action as

$$F^{\xi} = F - \delta F \quad \text{where} \quad \delta F = \mathrm{L}_{\xi} F = [\xi, F], \tag{6.1.9}$$

for any $\xi \in \mathsf{L}$. We say this the gauge transformation of F is *homogeneous* since the map $F \to F^{\xi}$ is $C^{\infty}(\mathcal{M})$-linear. This also implies that a null curvature vanishes on all the gauge orbit of ω. At the first order in ξ, we see that the geometric action of L is compatible with the geometric action on the ordinary connection 1-form in the sense that

$$F^{\xi} = \widehat{\mathrm{d}}\omega^{\xi} + \frac{1}{2}[\omega^{\xi}, \omega^{\xi}] \tag{6.1.10}$$

The geometric action of L on F preserves its L-horizontality in the sense that $F^{\xi}(\iota \circ \ell, \mathfrak{X}) = 0$, for any $\mathfrak{X} \in \mathsf{A}$ and $\xi, \ell \in \mathsf{L}$.

6.2 Geometric action of L on generalized connections

The geometric action of the kernel is applied on generalized connections on A. Problems arise as we consider the geometric gauge transformation of the induced ordinary connection ω. Indeed, this ordinary connection is associated to a background connection $\mathring{\omega}$ which, by definition, does not support the gauge action of L. To recover a convenient transformation for ω, we have to define a "non-geometric" gauge action of L *i.e.* an *algebraic gauge action*.

Let $\varpi \in \Omega^1(\mathsf{A}, \mathsf{L})$ be a generalized connection 1-form on A and $(\mathsf{L}, i, \mathrm{L})$ be the Cartan operation on A. The infinitesimal geometric action of L on ϖ is given by the Lie derivative so that we define $\varpi^{\xi} = \varpi - \delta\varpi$ with $\delta\varpi = \mathrm{L}_{\xi}\varpi = [\xi, \varpi(\mathfrak{X})] - \varpi \circ \iota([\xi, \mathfrak{X}])$. Then, we obtain

$$\varpi^{\xi} = \varpi - [\xi, \varpi] - \varpi(\iota \circ \widehat{\mathrm{d}}\xi) \tag{6.2.1}$$

We see that if ϖ is normalized on L, we obtain the gauge transformations (6.1.3).

With respect to the decomposition of ϖ as in formula (4.2.3), we compute the infinitesimal action of L on the reduced kernel endomorphism τ and on the induced ordinary connection ω. The background connection term used to define the ordinary connection is invariant under the action of L. Indeed, this background connection term is related to both the metric connection associated to a metric $\widehat{g}$ on A and the mixed local basis used to locally decompose differential forms. For these two reasons, it seems illegitimate to consider the background connection as a gauge field. Then we assume that the gauge action is not represented on it or, equivalently, that the representation of L is trivial on $\mathring{\omega}$.

We define the gauge transformation of the reduced kernel endomorphism as $\tau^{\xi} = \varpi^{\xi} \circ \iota + \mathrm{Id}_{\mathsf{L}}$ and the gauge transformation of the induced ordinary connection as $\omega^{\xi} = \varpi^{\xi} + \tau^{\xi}(\mathring{\omega})$. A straightforward computation leads to the transformations:

$$\begin{cases} \tau^{\xi}(\ell) = \tau(\ell) - [\xi, \tau(\ell)] - \tau([\ell, \xi]) \\ \omega^{\xi} = \omega - [\xi, \omega] + \widehat{\mathrm{d}}\xi - \alpha_{\xi}, \end{cases} \tag{6.2.2}$$

for any $\ell, \xi \in \mathsf{L}$, where we have defined $\alpha_{\xi} = \tau([\mathring{\Theta}, \xi]) \in \Omega^1(\mathsf{A}, \mathsf{L})$. This term is clearly a reminiscent of the fact that the background connection does not support the action of L. It is related to a geometric obstruction of the gauge action of L. Nevertheless, ω^{ξ} is still an ordinary connection 1-form since α_{ξ} is L-horizontal and then we obtain $\omega^{\xi}(\iota \circ \ell) = -\ell$ for any $\ell \in \mathsf{L}$. The gauge transformation of the covariant connection $\Theta : \mathsf{A} \to \mathsf{A}$ associated to ω is induced from ω^{ξ} as $\Theta^{\xi}(\mathfrak{X}) = \mathfrak{X} + \iota \circ \omega^{\xi}(\mathfrak{X})$,for any $\mathfrak{X} \in \mathsf{A}$. Then, one obtains $\Theta^{\xi} = \Theta - [\xi, \Theta] - \alpha_{\xi}$ for any $\xi \in \mathsf{L}$. Again, the 1-form α_{ξ} prevents the covariant connection Θ to transform as a geometric object.

The generalized covariant connection $\widehat{\Theta} : \mathsf{A} \to \mathsf{A}$ associated to ϖ defined in section 4.2.1 transforms as

$$\widehat{\Theta}^{\xi} = \widehat{\Theta} - \delta\widehat{\Theta} \quad \text{with} \quad \delta\widehat{\Theta} = \mathrm{L}_{\xi}\widehat{\Theta} = [\xi, \widehat{\Theta}] + \widehat{\Theta} \circ \iota(\widehat{\mathrm{d}}\xi). \tag{6.2.3}$$

Using a straightforward computation, the generalized covariant map $\widehat{\Theta}^\xi$ is related to the generalized connection ϖ^ξ as

$$\widehat{\Theta}^\xi(\mathfrak{X}) = \mathfrak{X} + \iota \circ \varpi^\xi(\mathfrak{X}). \tag{6.2.4}$$

Contrary to the covariant derivative associated to an ordinary connection 1-form, the covariant derivative associated to a generalized connection does not preserve the representation space with respect to the geometric action of L. To see this, consider an element $s \in \Gamma(\mathcal{E})$ which supports a geometric action of L, as in (6.1.5). For any $\xi \in \mathsf{L}$, the generalized covariant derivative of s transforms as $\widehat{\mathcal{D}}^\xi s^\xi = \phi(\widehat{\Theta}^\xi)s^\xi$. We compare this gauge transformation with $(\widehat{\mathcal{D}}s)^\xi = \widehat{\mathcal{D}}s - \phi_{\mathsf{L}}(\xi)\widehat{\mathcal{D}}s$ and we obtain, at first order in ξ, the relation

$$\widehat{\mathcal{D}}^\xi s^\xi = (\widehat{\mathcal{D}}s)^\xi - \phi(\widehat{\Theta} \circ \iota(\widehat{\mathrm{d}}\xi))s \tag{6.2.5}$$

The obstruction for the generalized covariant derivative associated to ϖ to preserve the representation space $\mathcal{E}$, with respect to the geometric infinitesimal gauge action of L is encoded in the reduced kernel endomorphism in the sense that

$$(\widehat{\mathcal{D}}s)^\xi - \widehat{\mathcal{D}}^\xi s^\xi = \phi_{\mathsf{L}}(\tau(\widehat{\mathrm{d}}\xi))s \tag{6.2.6}$$

For $\tau = 0$, the right-hand-term vanishes and the covariant derivative preserves $\Gamma(\mathcal{E})$ with respect to the action of L. Then, the incompatibility between these two representations of L comes from the algebraic component τ of ϖ. This will be also apparent in the gauge transformations of the generalized curvature associated to ϖ.

Let $\widehat{F} \in \Omega^2(\mathsf{A}, \mathsf{L})$ be the curvature associated to the generalized connection ϖ. This curvature supports two distinct representations of L. It can be either defined by the geometric gauge action of L as

$$\widehat{F}^\xi = \widehat{F} - \delta\widehat{F} \quad \text{with} \quad \delta\widehat{F} = \mathrm{L}_\xi \widehat{F}, \tag{6.2.7}$$

or it can be defined using the Cartan structure equation

$$\widehat{F}^\xi = \widehat{\mathrm{d}}\varpi^\xi + \tfrac{1}{2}[\varpi^\xi, \varpi^\xi], \tag{6.2.8}$$

for any $\xi \in \mathsf{L}$ where $\varpi^\xi = \varpi - \mathrm{L}_\xi\varpi$. Similarly to the generalized covariant derivatives, these two gauge transformations of the curvature $\widehat{F}$ are not compatible.

Consider the infinitesimal gauge transformation of $\widehat{F}$ induced by ϖ^ξ by the Cartan structure equation $\widehat{F}^\xi = \widehat{\mathrm{d}}\varpi^\xi + \frac{1}{2}[\varpi^\xi, \varpi^\xi]$. Directly, one shows that the three elements exhibited in the decomposition of $\widehat{F}$ (see section 4.2.3) transform respectively as

$$\begin{cases} \rho^*\widehat{R} & \mapsto \quad (\rho^*\widehat{R})^\xi = \rho^*\widehat{R} - [\xi, \rho^*\widehat{R}] \\ & \qquad\qquad +\tau([\mathring{F}, \xi]) - [\Theta, \alpha_\xi] \\ & \qquad\qquad +\mathcal{O}(\xi^2) \\ (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} & \mapsto \quad (\rho^*\mathcal{D}\tau)^\xi \circ \mathring{\omega} = (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} - [\xi, (\rho^*\mathcal{D}\tau)\circ\mathring{\omega}] \\ & \qquad\qquad -[\Theta, \tau([\mathring{\omega}, \xi])] + \tau\left([[\mathring{\Theta}, \mathring{\omega}], \xi]\right) - [\alpha(\xi), \tau(\mathring{\omega})] \\ & \qquad\qquad +\mathcal{O}(\xi^2) \\ \mathring{\omega}^* R_\tau & \mapsto \quad (\mathring{\omega}^* R_\tau)^\xi = \mathring{\omega}^* R_\tau - [\xi, \mathring{\omega}^* R_\tau] \\ & \qquad\qquad -\tfrac{1}{2}\tau([[\mathring{\omega}, \mathring{\omega}], \xi]) - [\tau(\mathring{\omega}), \tau([\mathring{\omega}, \xi])] \\ & \qquad\qquad +\mathcal{O}(\xi^2) \end{cases} \tag{6.2.9}$$

In the three cases, the first order linear term in ξ is decomposed into two parts. The former is given by the adjoint action of $\xi \in \mathsf{L}$ and corresponds to the homogeneous part of the transformation. The latter involves non homogeneous terms depending mainly on τ and α_ξ. These terms are badly interpretable since we know that this kind of gauge transformations do not come from any geometric structure, due to the presence of the background term ω. Still, we remark that gauge transformations preserve the geometric and the algebraic statues of each terms. In particular, the geometric action of L on the third term, which is the obstruction for τ to be a morphism of Lie algebras, preserves this algebraic status.

Now, consider the infinitesimal gauge transformation of $\widehat{F}$ given by the geometric action of L as $\widehat{F} \to \widehat{F}^\xi = \widehat{F} - \delta\widehat{F}$ where $\delta\widehat{F} = \mathrm{L}_\xi\widehat{F}$ for any $\xi \in \mathsf{L}$. By linearity, the geometric action of L is distributed on the three factors $\rho^*\widehat{R}$, $(\rho^*\mathcal{D}\tau)\circ\mathring{\omega}$ and $\mathring{\omega}^*R_\tau$ of the previous decomposition so that we obtain

$$\begin{cases} \rho^*R & \mapsto & (\rho^*R)^\xi = \rho^*R - \mathrm{L}_\xi(\rho^*R) \\ (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} & \mapsto & ((\rho^*\mathcal{D}\tau)\circ\mathring{\omega})^\xi = (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} - \mathrm{L}_\xi\left((\rho^*\mathcal{D}\tau)\circ\mathring{\omega}\right) \\ \mathring{\omega}^*R_\tau & \mapsto & (\mathring{\omega}^*R_\tau)^\xi = \mathring{\omega}^*R_\tau - \mathrm{L}_\xi\mathring{\omega}^*R_\tau \end{cases} \tag{6.2.10}$$

It results in the geometric action of L on each of these three terms

$$\begin{cases} \mathrm{L}_\xi(\rho^*R) & = & [\xi, \rho^*R] \\ \mathrm{L}_\xi\left((\rho^*\mathcal{D}\tau)\circ\mathring{\omega}\right) & = & [\xi, (\rho^*\mathcal{D}\tau)\circ\mathring{\omega}] - [\Theta, \tau(\widehat{\mathrm{d}}\xi)] + \tau([\mathring{\Theta}, \widehat{\mathrm{d}}\xi]) \\ \mathrm{L}_\xi(\mathring{\omega}^*R_\tau) & = & [\xi, \mathring{\omega}^*R_\tau] + \tau([\widehat{\mathrm{d}}\xi, \mathring{\omega}]) - [\tau(\widehat{\mathrm{d}}\xi), \tau(\mathring{\omega})] \end{cases} \tag{6.2.11}$$

Contrary to the previous type of gauge transformations, the geometric transformations of each terms do not preserve their respective geometric and algebraic statues.

We denote by $(\widehat{F}^\xi)_{\text{geo}}$ the geometric action of $\widehat{F}$ and we denote by $(\widehat{F}^\xi)_{\text{ind}}$ the gauge transformation $\widehat{F}$ induced by the gauge transformed connection ϖ^ξ. With the same notation, we make the distinction for the gauge transformations of the elements ρ^*R, $(\rho^*\mathcal{D}\tau)\circ\omega$ and $\mathring{\omega}^*R_\tau$. We establish the following relations:

$$\begin{aligned} &(\rho^*R)^\xi_{\text{ind}} - (\rho^*R)^\xi_{\text{geo}} = \tau([\mathring{F}, \xi]) - [\Theta, \alpha_\xi] \\ &(\rho^*\mathcal{D}\tau)^\xi_{\text{ind}} - (\rho^*\mathcal{D}\tau)^\xi_{\text{geo}} = -[\Theta, \alpha_\xi] + \tau([\mathring{\Theta}, \alpha_\xi]) - [\alpha_\xi, \tau(\mathring{\omega})] + \tau([[\mathring{\Theta}, \xi], \mathring{\omega}]) \\ &(\mathring{\omega}^*R_\tau)^\xi_{\text{ind}} - (\mathring{\omega}^*R_\tau)^\xi_{\text{geo}} = -\tau([[\mathring{\Theta}, \xi], \mathring{\omega}]) + [\alpha_\xi, \tau(\mathring{\omega})] \end{aligned} \tag{6.2.12}$$

The obstruction for these two types of gauge transformation of the curvature to be equivalently defined either as the geometric action of L on $\widehat{F}$ or by induction from the gauge transformed generalized connection ϖ^ξ is encoded in the reduced kernel endomorphism of ϖ. Indeed, we directly reads that these three obstructions vanishes for $\tau = 0$ *i.e.* when the space of generalized connections is restricted to the space of ordinary connections.

6.2.1 Gauge action on Atiyah Lie algebroids

In this section, we consider an Atiyah Lie algebroid associated to a principal bundle $\mathcal{P}(\mathcal{M}, G)$. Here, the geometric action of the kernel L can be substituted by the action of a gauge group $\mathcal{G}$. Then, the geometric action on $\Gamma_G(\mathcal{P}, \mathfrak{g})$ is given by the action of the group of vertical automorphisms. At first order, this corresponds exactly to the infinitesimal geometric action of $\Gamma_G(\mathcal{P}, \mathfrak{g})$.

In the theory of fiber bundles, the gauge action of the structure group is related to the space of vertical automorphisms $f : \mathcal{P}_p \to \mathcal{P}_p$, where $p \in \mathcal{M}$, such that $f(u{\cdot}g) = f(u){\cdot}g$, for any $u \in \mathcal{P}$ and $g \in G$. This application is equivalent to the data of a G-equivariant group valued field $g : \mathcal{P} \to G$ which fulfills $g(u{\cdot}a) = a^{-1}{\cdot}g(u){\cdot}a$, for any $a \in G$. We denote by $\mathcal{G}$ the set of these functions g and we call it the *gauge group* on $\mathcal{P}$. The maps $f : \mathcal{P}_p \to \mathcal{P}_p$ and $g : \mathcal{P} \to G$ are related by the formula $f(u) = u{\cdot}g(u)$. Assume G is connected and simply connected so that any element g can be written under the form $g = \exp(\lambda)$, where $\lambda \in \mathfrak{g}$. Then, any element $g \in \mathcal{G}$ can be written under the form $g(u) = \exp(\lambda(u))$, where $\lambda : \mathcal{P} \to \mathfrak{g}$ with $\lambda(u{\cdot}a) = a^{-1}{\cdot}\lambda(u){\cdot}a$, for any $u \in \mathcal{P}$ and $a \in G$. Then, the kernel $\Gamma_G(\mathcal{P}, \mathfrak{g})$ of $\Gamma_G(\mathcal{P})$ generates the gauge group of $\mathcal{P}$.

What follow are classical results of differential geometry obtained in the context of Atiyah Lie algebroids.

The action of the gauge group on the space of right-invariant vector fields $\Gamma_G(\mathcal{P})$ is given in terms of flows as

$$(\mathfrak{X}^g)_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \phi_{\mathfrak{X},t}(u{\cdot}g(u)){\cdot}g^{-1}\left(\phi_{\mathfrak{X},t}(u{\cdot}g(u))\right) = \mathfrak{X}_u - \iota \circ \Big((\mathfrak{X}{\cdot}g^{-1})g\Big)_u \tag{6.2.13}$$

for any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$ and $u \in \mathcal{P}$. Basic geometric computations shows that the inhomogeneous term $(\mathfrak{X}{\cdot}g^{-1})g$ is an element of $\Gamma_G(\mathcal{P}, \mathfrak{g})$. With abuse of notations, we extend the differential $\widehat{\mathrm{d}}$ to $\mathcal{G}$ but we keep in mind that the quantity $\widehat{\mathrm{d}}g$ makes no sense and only terms of the form $g^{-1}\widehat{\mathrm{d}}g$ or $(\widehat{\mathrm{d}}g)g^{-1}$ are correctly defined. The term $g^{-1}(\widehat{\mathrm{d}}g) \in \Omega^1_{\mathrm{Lie}}(\mathcal{P}, \mathfrak{g})$ is defined as $g^{-1}(\widehat{\mathrm{d}}g)(\mathfrak{X}) = g^{-1}(\mathfrak{X}{\cdot}g)$, for any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$, same thing for $(\widehat{\mathrm{d}}g)g^{-1}$. It is direct to prove that $g^{-1}(\widehat{\mathrm{d}}g) + (\widehat{\mathrm{d}}g^{-1})g = 0$ and $g^{-1}(\widehat{\mathrm{d}}g)(\iota \circ \ell) = g^{-1}{\cdot}\ell{\cdot}g - \ell$, for any $\ell \in \Gamma_G(\mathcal{P}, \mathfrak{g})$.

The action of $\mathcal{G}$ on the elements $v \in \Gamma(\mathcal{P}, \mathfrak{g})$ is defined by the following commutating diagram

$$\begin{tikzcd}
\Gamma_G(\mathcal{P}) \arrow[r, "\mathcal{G}_*"] & \Gamma_G(\mathcal{P}) \\
\Gamma(\mathcal{P}, \mathfrak{g}) \arrow[u, "\iota"] \arrow[r, "\mathcal{G}_*"] & \Gamma(\mathcal{P}, \mathfrak{g}) \arrow[u, "\iota"']
\end{tikzcd}$$

where $\mathcal{G}_*$ denotes the geometric action of $\mathcal{G}$ on either $\Gamma_G(\mathcal{P})$ or on $\Gamma_G(\mathcal{P}, \mathfrak{g})$. Directly, one reads $((\iota \circ v)^g)_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} u{\cdot}e^{-tv(u)}{\cdot}g^{-1}\left(u{\cdot}e^{-tv(u)}\right){\cdot}g(u) = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} u{\cdot}g^{-1}(u){\cdot}e^{-tv(u)}{\cdot}g(u)$. One compares this result to $(\iota \circ v^g)_u = \frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} ue^{-tv^g(u)}$ so that $\mathcal{G}$ acts on $\Gamma_G(\mathcal{P}, \mathfrak{g})$ as $v^g = g^{-1}{\cdot}v{\cdot}g$, for any $g \in \mathcal{G}$. One defines the induced action of $\mathcal{G}$ on an ordinary connection 1-form so that this second diagram commutes

$$\begin{tikzcd}
\Gamma_G(\mathcal{P}) \arrow[r, "\mathcal{G}_*"] \arrow[d, "\omega"'] & \Gamma_G(\mathcal{P}) \arrow[d, "\omega^g"] \\
\Gamma(\mathcal{P}, \mathfrak{g}) \arrow[r, "\mathcal{G}_*"] & \Gamma(\mathcal{P}, \mathfrak{g})
\end{tikzcd}$$

This gives the relation : $(\omega(X))^g = \omega^g(X^g)$ for any $\mathfrak{X} \in \Gamma_G(\mathcal{P})$. It results in the gauge transformation of the ordinary connection 1-form ω as $\omega^g(X) = g^{-1}{\cdot}\omega(X - \iota((X{\cdot}g)g^{-1})){\cdot}g = g^{-1}{\cdot}\omega(X){\cdot}g + g^{-1}(X \cdot g)$ or, in a more compact notation $\omega^g = g^{-1}{\cdot}\omega{\cdot}g + g^{-1}{\cdot}\widehat{\mathrm{d}}g$. In the case of the matrix group, the element $g(u)$ can be written as $\exp(t\xi(u)) = 1 + t\xi(u) + \mathcal{O}(t^2)$. Making the derivation at $t = 0$, one obtains the formula (6.1.3).

We consider a generalized connection ϖ on A. The definition of the geometric action of the gauge group $\mathcal{G}$ is given by the sme previous computations so that we obtain

$$\varpi^g = g^{-1}{\cdot}\varpi{\cdot}g - g^{-1}{\cdot}\varpi \circ \iota((\widehat{\mathrm{d}}g){\cdot}g^{-1}){\cdot}g. \tag{6.2.14}$$

With this definition, we induce the gauge transformation of the induced ordinary connection ω and the reduced kernel endomorphism τ as in (4.2.3). One defines the gauge transformed reduced kernel endomorphism as $\tau^g = \varpi^g \circ \iota + \mathrm{Id}_{\mathsf{L}} : \mathsf{L} \to \mathsf{L}$ and the gauge transformed of the induced ordinary connection ω^g as $\omega^g = \varpi^g + \tau^g(\mathring{\omega})$. Here again, the background connection $\mathring{\omega}$ does not support the representation of the gauge group. A straightforward computation leads to the transformations:

$$\begin{cases} \tau^g(\ell) = g^{-1}\tau(g\ell g^{-1})g \\ \omega^g = g^{-1}{\cdot}\omega{\cdot}g + g^{-1}\widehat{\mathrm{d}}g - \alpha_g \end{cases} \tag{6.2.15}$$

for $\ell \in \mathsf{L}$ and $g \in \mathcal{G}$, where we have defined $\alpha_g = g^{-1}{\cdot}\tau(\mathring{\omega} - g{\cdot}\mathring{\omega}{\cdot}g^{-1} - g\widehat{\mathrm{d}}g^{-1})g$. This last term is reminiscent of the fact that the background connection does not support the action of $\mathcal{G}$. It corresponds to the "global" version of α_ξ. As in the infinitesimal case, ω^g is still an ordinary connection since α_g vanishes on L. Note that the gauge transformation of the reduced kernel endormophism τ associated to ϖ can be obtained by the geometric action of $\mathcal{G}$. To see this, one considers the following commutative diagram:

$$\begin{tikzcd} \Gamma_G(\mathcal{P}, \mathfrak{g}) \arrow[r, "G_*"] \arrow[d, "\tau"'] & \Gamma_G(\mathcal{P}, \mathfrak{g}) \arrow[d, "\tau^g"] \\ \Gamma(\mathcal{P}, \mathfrak{g}) \arrow[r, "G_*"] & \Gamma(\mathcal{P}, \mathfrak{g}) \end{tikzcd}$$

Then, we obtain $\tau^g(v^g) = (\tau(v))^g$ that we can write $\tau^g(v) = g^{-1}\tau(gvg^{-1})g$, for any $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$ and $g \in \mathcal{G}$.

To finish with the computation of the gauge group on Atiyah Lie algebroids, we explicitly compute the action of $\mathcal{G}$ on the three elements exhibited in (4.2.14). Then, the gauge transformation of $\widehat{F}$ is defined as $\widehat{F}^g = \widehat{\mathrm{d}}\varpi^g + \frac{1}{2}[\varpi^g, \varpi^g]$. We obtain

$$\begin{cases} \rho^*\widehat{R} & \mapsto & (\rho^*\widehat{R})^g = g^{-1}(\rho^*\widehat{R}){\cdot}g \\ & & \qquad -\widehat{\mathrm{d}}(\alpha_g) - [\omega^g, \alpha_g] + \frac{1}{2}[\alpha_g, \alpha_g] \\ (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} & \mapsto & (\rho^*\mathcal{D}\tau)^g \circ \mathring{\omega} = g^{-1}{\cdot}[\Theta, \tau(g{\cdot}\mathring{\omega}{\cdot}g^{-1})]{\cdot}g - g^{-1}{\cdot}\tau(g{\cdot}[\mathring{\Theta}, \mathring{\omega}]{\cdot}g^{-1}){\cdot}g \\ & & \qquad -[\alpha_g, g^{-1}\tau(g\mathring{\omega}g^{-1})g^{-1}] \\ \mathring{\omega}^*R_\tau & \mapsto & (\mathring{\omega}^*R_\tau)^g = g^{-1}{\cdot}\left(\tau(g[\mathring{\omega}, \mathring{\omega}]g^{-1})g - [\tau(g\mathring{\omega}g^{-1}), \tau(g\mathring{\omega}g^{-1})]\right){\cdot}g \end{cases} \tag{6.2.16}$$

Note that the action of $\mathcal{G}$ on the third term, which is the obstruction for τ to be a morphism of Lie algebra, preserves this status.

6.3 Algebraic action of L

In the previous section, we have shown that the geometric action of L is related to the infinitesimal version of the action of the gauge group. In the context of ordinary connections

ω, this action of L is compatible with the geometric nature of the objects derived from ω such as the covariant derivative, the curvature, etc. However, the "hybrid" nature of generalized connection is in conflict with the "pure geometric" action of L. This is obvious in the formulas of section 6.2 where the reduced kernel endomorphism τ, which measures the non-geometric nature of ϖ, is involved in all the "undesirable terms" of the gauge transformations. The most problematic feature related to the geometric gauge action on generalized connections concerns the covariant derivative. Indeed, the geometric action of L does not preserve the "covariance" of $\widehat{\Theta}$.

The algebraic gauge action of L is introduced to restore a convenient gauge transformation with respect to the covariant derivative associated to ϖ. This requirement leads to impose a new transformation for ϖ. As we will see, the geometric and the algebraic action of L coincide for $\tau = 0$. For $\tau \neq 0$, the algebraic action of L cannot be interpreted as geometric construction: this explains the terminology "algebraic gauge action".

With this choice of gauge action, we induce the corresponding gauge transformations of the generalized covariant connection, the curvature, *etc.* Then, we obtain homogeneous transformations which will be used to construct gauge field theories based on generalized connections.

Let $\mathcal{E}$ be a representation space of A. We assume that the kernel L acts infinitesimally on $s \in \Gamma(\mathcal{E})$ as $s^{\xi} = s - \phi_{\mathsf{L}}(\xi)s$, where ϕ_{L} denote the representation of L on $\mathcal{E}$, as in the previous case. This gauge transformation corresponds to the most natural representation of L on the "matter fields" of the theory. In the case of Atiyah Lie algebroids, the fiber bundle $\mathcal{E}$ is a vector bundle associated to $\mathcal{P}$, with fiber F, equipped with the left action $\ell : G \to \mathrm{End}(F)$. The gauge group $\mathcal{G}$ acts on $s : \mathcal{P} \to F$ as $s^g = \ell(g^{-1})s$, for any $g \in \mathcal{G}$.

Let ϖ be a generalized connection on A and $\widehat{\mathcal{D}}_{\phi} = \phi(\widehat{\Theta}) : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$ the covariant derivative associated to ϖ. One asks for the covariant derivative $\widehat{\mathcal{D}}_{\phi}$ to have homogeneous gauge transformations either with respect to the infinitesimal action of L or, if it exists, with respect to the action of $\mathcal{G}$. Then, we impose

$$\widehat{\mathcal{D}}_{\phi}(\mathfrak{X})^{\xi} s^{\xi} = \widehat{\mathcal{D}}_{\phi}(\mathfrak{X})s - \phi_{\mathsf{L}}(\xi)\widehat{\mathcal{D}}_{\phi}(\mathfrak{X})s \quad \text{or} \quad \widehat{\mathcal{D}}_{\phi}(\mathfrak{X})^{g} s^{g} = \ell(g^{-1})\widehat{\mathcal{D}}_{\phi}(\mathfrak{X})s, \tag{6.3.1}$$

for any $\mathfrak{X} \in \mathsf{A}$ and $s \in \Gamma(\mathcal{E})$. These two formulas are equivalent, up to the first order in ξ. Then, with respect to the action of L, respectively $\mathcal{G}$, the generalized connection ϖ is required to transform as

$$\varpi^{\xi} = \varpi - [\xi, \varpi] + \widehat{\mathrm{d}}\xi \quad \text{or} \quad \varpi^{g} = g^{-1}\varpi g + g^{-1}\widehat{\mathrm{d}}g, \tag{6.3.2}$$

where $g^{-1}\widehat{\mathrm{d}}g$ has been defined in 6.2.1.

These two formulas differ from the formulas (6.2.1) and (6.2.14) obtained from the geometric action of L or $\mathcal{G}$. This is an important point: the fact that the generalized covariant derivative preserves the gauge action of L (or $\mathcal{G}$) on $\mathcal{E}$ leads to a non-geometrical action on the generalized connection ϖ. Formulas in (6.3.2) give the *infinitesimal algebraic action of* L and the *algebraic action of* $\mathcal{G}$ on ϖ, respectively.

We see now how this algebraic action of L is induced, starting from the transformation of ϖ to its geometric component ω and its algebraic component τ of ϖ. One recalls that the background connection, by definition, doesn't support the action of L (neither geometric nor algebraic). With respect to the decomposition of ϖ into (ω, τ) (*c.f.* section 4.2.1), the computation of the infinitesimal algebraic action of L on the induced ordinary connection ω and the reduced kernel endomorphism τ reads

$$\begin{cases} \omega^{\xi} = \omega - [\xi, \omega] + \widehat{\mathrm{d}}\xi \\ \tau^{\xi}(\ell) = \tau(\ell) - [\xi, \tau(\ell)] \end{cases} \tag{6.3.3}$$

for any $\ell, \xi \in \mathsf{L}$. With respect to the algebraic action of L, the induced ordinary connection ω transforms like a true ordinary connection on A. Contrary to the geometric transformation, the ad representation of $\mathfrak{g}$ acts only on the target space of the τ and no more on its source space. Similarly, for $\ell \in \mathsf{L}$ and $g \in \mathcal{G}$, the algebraic action of $\mathcal{G}$ on the induced ordinary connection and the reduced kernel endomorphism gives the following transformations

$$\begin{cases} \omega^{g} = g^{-1}\omega g + g^{-1}\widehat{\mathrm{d}}g \\ \tau^{g}(\ell) = g^{-1}\tau(\ell) g \end{cases} \tag{6.3.4}$$

These two transformations are the global version of the infinitesimal algebraic action of L.

The distinction between the geometric and the algebraic gauge actions of L is encoded in the reduced kernel endomorphism associated to ϖ. A direct computation highlights this assertion. Let $\varpi^{\xi}_{\mathrm{alg}}$ be the algebraic action of ξ on ϖ and let $\varpi^{\xi}_{\mathrm{geo}}$ be the geometric action of ξ on ϖ. The difference between these two representations gives $\varpi^{\xi}_{\mathrm{alg}} - \varpi^{\xi}_{\mathrm{geo}} = \tau(\widehat{\mathrm{d}}\xi)$ for any $\xi \in \mathsf{L}$. With respect to the geometric and algebraic action of $g \in \mathcal{G}$, one obtains $\varpi^{g}_{\mathrm{alg}} - \varpi^{g}_{\mathrm{geo}} = \tau(g^{-1}\widehat{\mathrm{d}}g)$ for any $g \in \mathcal{G}$. Then, when restricted to the subspace of ordinary connections, the geometric action and the algebraic action of L or of the gauge group $\mathcal{G}$ coincide. Out of the space of ordinary connections, this could not happen except if, by a process of symmetry reduction, the gauge parameter ξ (or the Lie algebra of the reduced group of symmetry) is restricted to $\ker(\tau)$. This point is out of the scope of the present PhD thesis.

We close this section by the definition of the algebraic action of L and $\mathcal{G}$ on the curvature associated to ϖ. Consider the algebraic action of L on ϖ, the induced infinitesimal transformation of the curvature $\widehat{F}$ associated to ϖ is induced from the expression of ϖ^{ξ} so that it transforms as $\widehat{F}^{\xi} = \widehat{F} - [\xi, \widehat{F}]$. Then, the three components associated to the decomposition of $\widehat{F}$ with respect to their geometric and algebraic status transform respectively as:

$$\begin{cases} \rho^*\widehat{R} & \mapsto & (\rho^*\widehat{R})^{\xi} = \rho^*\widehat{R} - [\xi, \rho^*\widehat{R}] \\ (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} & \mapsto & (\rho^*\mathcal{D}\tau)^{\xi}\circ\mathring{\omega} = (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} - [\xi, (\rho^*\mathcal{D}\tau)\circ\mathring{\omega}] \\ \mathring{\omega}^*R_{\tau} & \mapsto & (\mathring{\omega}^*R_{\tau})^{\xi} = \mathring{\omega}^*R_{\tau} - [\xi, \mathring{\omega}^*R_{\tau}] \end{cases} \tag{6.3.5}$$

One obtains homogeneous gauge transformations. In the three cases, the first order linear term is given by the adjoint action of $\xi \in \mathsf{L}$. Note that the action of L on the third term, which is the obstruction for τ to be a morphism of Lie algebras, preserves this status. Similarly, consider the algebraic representation of $\mathcal{G}$ on ϖ, the induced gauge transformation of the curvature $\widehat{F}$ associated to ϖ is given by $\widehat{F}^{g} = g^{-1}\widehat{F}g$ for any $g \in \mathcal{G}$. Then, the three components defined in (4.2.14) transform as:

$$\begin{cases} \rho^*\widehat{R} & \mapsto & (\rho^*\widehat{R})^{g} = g^{-1}(\rho^*\widehat{R})g \\ (\rho^*\mathcal{D}\tau)\circ\mathring{\omega} & \mapsto & (\rho^*\mathcal{D}\tau)^{g}\circ\mathring{\omega} = g^{-1}(\rho^*\mathcal{D}\tau)g \\ \mathring{\omega}^*R_{\tau} & \mapsto & (\mathring{\omega}^*R_{\tau})^{g} = g^{-1}(\mathring{\omega}^*R_{\tau})g \end{cases} \tag{6.3.6}$$

In the three cases, the gauge transformation is given by the adjoint action of $g \in \mathcal{G}$. In the next chapter, we will consider only algebraic gauge transformations.

Chapter 7

Yang-Mills-Higgs type theories

The mathematical description of the SM of particle physics is based on the existence of both some *inner symmetries* and the so-called *gauge principle* which claims that any observable in physics have to be invariant with respect to the action of the corresponding symmetry group. From these two fundamental concepts, one induces that the theory of fiber bundles is an adapted mathematical framework to describe these inner symmetries and that physical quantities related to inner degrees of freedom have to be gauge-invariant quantities.

As a direct application of this observation, the YM theories are geometric constructions which result in the dynamic of the gauge bosons of the SM. These particles mediate the fundamental electromagnetic and nuclear interactions and they can be described in terms of geometric objects defined on a trivial principal bundle $\mathcal{P}(\mathcal{M}, G)$. In this description, the gauge group $\mathcal{G}$ of the theory is related to the group of vertical automorphism on $\mathcal{P}$ and the minimal coupling term between matter fields and gauge bosons is given by the covariant derivative associated to a connection on $\mathcal{P}$. To construct gauge invariant quantities from geometric objects, YM theories considers the curvature R of this connection, which is the simplest first order derivative object with homogeneous gauge transformations. Then, it results in the definition of a gauge invariant action functional written as the "norm" of R. For $U(1)$-gauge theories, this theory gives exactly the Maxwell equations of the electromagnetic theory. This construction is detailed in [Nak03; TS87]. YM models are given by the *theory of connections* on $\mathcal{P}$.

Then, this description establishes a correspondence between geometric objects and the physical content of the SM: gauge bosons A_μ are local trivializations of connections ω on $\mathcal{P}$, the field strength $F_{\mu\nu}$ are related to the curvature of ω, etc. Upon the gauge field theories based on the theory of connections, additional structures on $\mathcal{P}$ are required to take also into account the description of matter fields, *e.g.* scalar fields and spinor fields which are described using associated vector bundles or Dirac bundles, respectively.

In the Lagrangian of the SM, the mass term for gauge bosons A_μ are given under the form $m_{ab}A^a_\mu A^{\mu\, b}$, where the summation is made over the Latin indices $a, b = 1, \dots, \dim(G)$ where m denotes the matrix of mass associated to the gauge bosons. However, it is not possible to add such terms in the theory without "breaking" the global gauge invariance of the Lagrangian. Thus, YM models based on the theory of connections are not suitable to describe massive vectors bosons $W^\pm_\mu$ and Z^0_μ, related to the weak interaction, whose mass have been experimentally measured to be 80.38 ± 0.015 GeV and 91.1876 ± 0.0021 GeV, respectively.

The mechanism used to restore the compatibility between YM gauge field theories and massive vector bosons is provided by the Brout-Englert-Higgs-Hagen-Guralnik-Kibble (BEHHGK, pronouned "beck") mechanism of spontaneous symmetry breaking [IZ85; PS95]. To perform this mechanism, additional structures have to be implemented on YM models.

It requires some *added-by-hand* objects such as a scalar field ϕ coupled to gauge bosons A_μ, which is embedded into a quartic potential. This potential is adjusted with dynamical parameters which depend on the energy scale of the system. For low energies, the field ϕ realizes a spontaneous polarization into the so-called vacuum configuration corresponding to a minimum of its potential. This mechanism is purely analog to the spontaneous polarization of magnetic crystals in ferro-magnetic theories. From the polarization of ϕ, the symmetry group G is *reduced* to one of its subgroup. Then, the mass term $m_{ab}A^a_\mu A^{\mu\, b}$ becomes gauge invariant and the gauge bosons A_μ acquire mass. Applied to the electroweak sector of the SM, this mechanism gives correctly the $U(1)$-charged current $W^+_\mu W^{\mu\,-}$ and the neutral current $Z^0_\mu Z^{0\,\mu}$ with the correct ratio of masses.

Then, the scalar field ϕ is added-by-hand in the theory as a new particle of the SM. Same thing for the potential, it has been added *a posteriori* to permit the polarization of ϕ. Thus, the spontaneous symmetry breaking mechanism is *out of the geometry of connections*. This means that the mass of gauge bosons does not come from intrinsic geometric objects but, instead, from the interaction with an external field.

The objective of this chapter is to establish a mathematical construction based on transitive Lie algebroids in order to obtain, at the end, a YMH theory strictly defined in the geometry of (generalized) connections in the sense that it will be simply constructed as the "norm" of the curvature associated to ϖ (the explicit expression is given in (7.1.1)). It results in that the space of generalized connections provide the algebraic extension that will permit to obtain a gauge theory greatly analog with the BEHHGK mechanism of symmetry reduction. However, this model presents some deep distinctions with the usual scheme. In particular, in our construction, the scalar field ϕ is substituted by the reduced kernel endomorphism $\tau : \mathsf{L} \to \mathsf{L}$ associated to a generalized connection ϖ. It results in a gauge field theory which is decomposed as the sum of a pure YM theory, a covariant derivative of the field τ^b_a and a potential term for τ^b_a. Moreover, this potential is not related to dynamical parameters and instead, it is interpreted as an obstruction for τ to be a morphism of Lie algebras. Thus, we will define the space of *solutions* associated to this theory. A first (trivial) solution is given by $\tau = 0$, which gives the usual YM theory. A second one (less trivial) is given by $\tau^b_a = \delta^b_a$ which gives a YM theory with massive vector fields.

In this chapter, all the constructions of the previous chapters are used to properly define a gauge invariant theory based on generalized connections on transitive Lie algebroids. In particular, we consider differential calculi on transitive Lie algebroids and systems of local trivializations. We also use an inner non-degenerate metric $\widehat{g}$ and the orientation on A to construct a Hodge star operator and a scalar product on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. The infinitesimal action of the gauge group is given by the algebraic action of L. It results in a gauge invariant functional action depending only on ϖ and defined as the square of $\widehat{F}$. Straightforward computations lead to the explicit expression of the theory in terms of gauge fields defined on spacetime. Discussions and comments will highlight the relevance of this construction.

7.1 Gauge field theories using generalized connections on transitive Lie algebroids

The gauge invariant action functional $\mathcal{S}_{Gauge}[\varpi]$ based on generalized connection is defined under a compact form so that it is direct to check that this quantity is invariant with respect to the algebraic action of L. Once we are convinced of this, we can give a more explicit formulation of $\mathcal{S}_{Gauge}[\varpi]$, with the guarantee that the obtained results won't lost its gauge-invariance with respect to L. The objective of this first section is, starting from the formal definition of this gauge action, to give the explicit expression of the associated Lagrangian. Under its final form, the Lagrangian depends only on Greek and Latin indexes coming from the local trivialization of transitive Lie algebroids.

7.1.1 Gauge invariant action functional

Let $\mathsf{A} \xrightarrow{\rho} \mathcal{M}$ be an orientable transitive Lie algebroid over $\mathcal{M}$. It is equipped with a non-degenerate and inner-non degenerate metric $\widehat{g} = (g, h, \mathring{\nabla})$ on A with g a non-degenerate metric on $\mathcal{M}$ and h a locally constant Killing inner (non-degenerate) metric on L. It is also equipped with an atlas of Lie algebroids $(\mathcal{U}_i, S_i)_{i\in I} = (\mathcal{U}_i, \Psi_i, \nabla_i^0)_{i\in I}$ which permits to realize local isomorphisms of differential complexes between $\Omega^\bullet(\mathsf{A}, \mathsf{L})$ and $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$.

Let $\widehat{F} \in \Omega^2(\mathsf{A}, \mathsf{L})$ be the curvature associated to the generalized connection $\varpi \in \Omega^1(\mathsf{A}, \mathsf{L})$. The *functional action associated to the generalized connection* ϖ on A is a scalar quantity which is invariant with respect to the infinitesimal algebraic action of L as depicted in section 6.3. It depends only on ϖ and is defined as:

$$\mathcal{S}_{Gauge}[\varpi] = \langle \widehat{F}, \widehat{F} \rangle_h \tag{7.1.1}$$

We show that the action $\mathcal{S}_{Gauge}[\varpi]$ is invariant with respect to the algebraic action of L *i.e.* it transforms with respect to $\xi \in \mathsf{L}$ as $S^\xi = S - \delta S$ with $\delta\mathcal{S} = 0$. The demonstration of this gauge invariance is tedious, but with no mystery, so that we rather use some formal arguments than direct computations. Before starting, we convince ourselves of the two following facts. Firstly, the algebraic action of L is represented only on the target space of $\widehat{F}$ by the relation $\widehat{F}^\xi = \widehat{F} + \delta\widehat{F}$, with $\delta\widehat{F} = -[\xi, \widehat{F}]$, for any $\xi \in \mathsf{L}$ where $[\cdot,\cdot]$ is the graded Lie bracket on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. Secondly, the Hodge star product $\star$ acts only on the arguments of $\widehat{F}$, *i.e.* on the source space, leaving invariant the target space $\mathfrak{g}$. This is obvious in formula (5.3.2). Then, the Hodge star operator and the gauge action commute with the gauge action of L so that we obtain

$$\star(\widehat{F}^\xi) = \star(\widehat{F} - [\xi, \widehat{F}]) = \star\widehat{F} - [\xi, \star\widehat{F}] = (\star\widehat{F})^\xi \tag{7.1.2}$$

for any $\xi \in \mathsf{L}$. Now, we use the fact that the inner metric h is a Killing metric as defined in section 5.1.1 in order to obtain

$$\begin{aligned}
\langle \widehat{F}^\xi, \widehat{F}^\xi \rangle_h &:= \int_{\mathsf{A}} h(\widehat{F}^\xi, (\star\widehat{F}^\xi)) \\
&= \int_{\mathsf{A}} h(\widehat{F}^\xi, (\star\widehat{F})^\xi) \\
&= \int_{\mathsf{A}} h(\widehat{F}, \star\widehat{F}) + h(-[\xi, \widehat{F}], \star\widehat{F}) + h(\widehat{F}, -[\xi, \star\widehat{F}]) \\
&= \int_{\mathsf{A}} h(\widehat{F}, \star\widehat{F}) \\
&= \langle \widehat{F}, \widehat{F} \rangle_h.
\end{aligned}$$

Then, the action functional $\mathcal{S}_{Gauge}[\varpi]$ is gauge invariant with respect to the algebraic action of L. In the case of Atiyah Lie algebroids associated to $\mathcal{P}(\mathcal{M}, G)$. The gauge group $\mathcal{G}$ acts globally on the gauge fields of the theory so that, instead of the Killing metric h, we use an ad-invariant polynomial form of degrees 2. Then, using the same arguments and computations as before, we shows that the functional action associated to the generalized connection $\varpi \in \Omega^1_{\text{Lie}}(\mathcal{P}, \mathfrak{g})$ is invariant with respect of the algebraic action of $\mathcal{G}$.

7.1.2 Lagrangian density

In gauge field theories, one prefers use the Lagrangians rather than action functionals for the description of fields. In the context of gauge field theories based on transitive Lie algebroids, the Lagrangian density is well-defined and permits to go further in the description of the theory.

The *Lagrangian density* associated to the gauge invariant action functional $\mathcal{S}_{Gauge}[\varpi]$ is defined as

$$\mathcal{L}[\varpi]\text{dvol} = \int_{\text{inner}} h(\widehat{F}, \star\widehat{F}) \in C^\infty(\mathcal{M}) \tag{7.1.3}$$

so that $\mathcal{S}_{Gauge}[\varpi] = \int_{\mathcal{M}} \mathcal{L}[\varpi]\text{dvol}$, where dvol denotes the invariant volume form on $\mathcal{M}$. The inner integration operator gives a function defined on $\mathcal{M}$ which is independent of any system of local trivializations of A. From the gauge invariance of $\mathcal{S}_{Gauge}[\varpi]$, the Lagrangian density is also gauge-invariant.

We use the connection 1-form $\mathring{\omega}$ associated to the metric connection $\mathring{\nabla}$ to locally decompose $\widehat{F}$ in the mixed local basis as in (4.2.21). The Hodge star operator acts on $\widehat{F}$ as in definition (5.3.2), so that a straightforward computation gives

$$\begin{aligned}
h(\widehat{F}, \star\widehat{F}) =& \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\, h(\widehat{F}_{\mu_1\nu_1}, \widehat{F}_{\mu_2\nu_2})\, \epsilon_{\rho_1\dots\rho_m}\epsilon_{a_1\dots a_n}\, g^{\mu_2\rho_1}g^{\nu_2\rho_2} \\
&\qquad \mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\nu_1}\wedge \mathrm{d}x^{\rho_3}\wedge\dots\wedge \mathrm{d}x^{\rho_m}\otimes \mathring{\omega}^{a_1}_{\text{loc}}\wedge\dots\wedge\mathring{\omega}^{a_n}_{\text{loc}} \\
&+ \sqrt{\det(h)}\sqrt{\det(g)}\, h(\widehat{F}_{\mu_1 a_1}, \widehat{F}_{\mu_2 a_2})\, \epsilon_{\rho_1\dots\rho_m}\epsilon_{b_1\dots b_n}\, g^{\mu_2\rho_1}h^{a_2 b_1} \\
&\qquad \mathrm{d}x^{\mu_1}\wedge \mathrm{d}x^{\rho_2}\wedge\dots\wedge \mathrm{d}x^{\rho_m}\otimes \mathring{\omega}^{a_1}_{\text{loc}}\wedge\mathring{\omega}^{b_2}_{\text{loc}}\wedge\dots\wedge\mathring{\omega}^{b_n}_{\text{loc}} \\
&+ \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\, h(\widehat{F}_{a_1 b_1}, \widehat{F}_{a_2 b_2})\, \epsilon_{\rho_1\dots\rho_m}\epsilon_{c_1\dots c_n}\, h^{a_2 c_1}h^{b_2 c_2} \\
&\qquad \mathrm{d}x^{\rho_1}\wedge\dots\wedge \mathrm{d}x^{\rho_m}\otimes \mathring{\omega}^{a_1}_{\text{loc}}\wedge\mathring{\omega}^{b_1}_{\text{loc}}\wedge\mathring{\omega}^{c_3}_{\text{loc}}\wedge\dots\wedge\mathring{\omega}^{c_n}_{\text{loc}}
\end{aligned}$$

where $\epsilon_{\rho_1\dots\rho_m}$ and $\epsilon_{a_1\dots a_n}$ are the Levi-Civita tensors and $\widehat{F}_{\mu\nu}$, $\widehat{F}_{\mu a}$ and $\widehat{F}_{ab}$ are elements of $C^\infty(\mathcal{U})\otimes\mathfrak{g}$ corresponding to the local trivialization of $\widehat{F}$ over $\mathcal{U}$. By counting the number of basis $\mathrm{d}x^\mu$ and $\mathring{\omega}^a_{\text{loc}}$ on each term, we see that $h(\widehat{F}, \star\widehat{F})$ is $(m+n)$-differential form defined on $\mathsf{TLA}(\mathcal{U},\mathfrak{g})$ with values in $C^\infty(\mathcal{U})$. Moreover, this $(m+n)$-form is given by the squares, terms by terms, of the three components of $\widehat{F}_{\text{loc}}$ so that the geometric terms $\widehat{F}_{\mu\nu}$, the "mix" terms $\widehat{F}_{\mu a}$ and the algebraic terms $\widehat{F}_{ab}$ do not interfere which each others.

In order to use the scalar product defined on $\Omega^\bullet(\mathsf{A})$, the $(m+n)$-differential form $h(\widehat{F}, \star\widehat{F})$ has to be written in terms of the volume form ω_{vol} on A, as defined in (5.2.4), and the volume form dvol on $\mathcal{M}$. To this purpose, we use cumbersome combinatorial computations. It results of these combinations an elegant expression of $\mathcal{L}[\omega]$ given simply in terms of $\widehat{F}_{\mu\nu}$, $\widehat{F}_{\mu a}$ and $\widehat{F}_{ab}$. The expression of this Lagrangian in terms of the local gauge fields A_μ, $\mathring{A}_\mu$ and τ^b_a is given in section 7.1.3.

For practical reasons, the three "square" terms of $h(\widehat{F}, \star\widehat{F})$ are computed independently. The objective of each computation is to obtain a decomposition of the corresponding component on the basis $\mathrm{dvol} \otimes \mathring{\omega}_{\mathrm{vol}} \in \Omega^{m+n}(\mathsf{A})$. The first term gives

$$\frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1\nu_1}, \widehat{F}_{\mu_2\nu_2})\; \epsilon_{\rho_1\dots\rho_m}\epsilon_{a_1\dots a_n}\; g^{\mu_2\rho_1}g^{\nu_2\rho_2}$$

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\nu_1} \wedge \mathrm{d}x^{\rho_3} \wedge \dots \wedge \mathrm{d}x^{\rho_m} \wedge \mathring{\omega}^{a_1}_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^{a_n}_{\mathrm{loc}}$$

$$= \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1\nu_1}, \widehat{F}_{\mu_2\nu_2})\; \epsilon_{\rho_1\dots\rho_m}\epsilon_{a_1\dots a_n}\; g^{\mu_2\rho_1}g^{\nu_2\rho_2}$$

$$\frac{1}{m!n!}\epsilon^{\mu_1\nu_1\rho_3\dots\rho_m}\epsilon^{a_1\dots a_n}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\delta^{\mu_1\nu_1}_{\rho_1\rho_2}\widehat{F}_{\mu_1\nu_1}, \widehat{F}_{\mu_2\nu_2})\; g^{\mu_2\rho_1}g^{\nu_2\rho_2}$$

$$\frac{(m-2)!}{m!}\; \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \frac{(m-2)!}{m!}\; \sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\rho_1\rho_2}, \widehat{F}_{\mu_2\nu_2})\; g^{\mu_2\rho_1}g^{\nu_2\rho_2}$$

$$\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \frac{1}{m(m-1)}\; h(\widehat{F}_{\rho_1\rho_2}, \widehat{F}_{\mu_2\nu_2})\; g^{\mu_2\rho_1}g^{\nu_2\rho_2}\; \mathrm{dvol} \wedge \omega_{\mathrm{vol}} \tag{7.1.4}$$

where $\mathrm{dvol} = \sqrt{\det(g)}\; \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m$ is the volume form on $\mathcal{M}$ and $\omega_{\mathrm{vol}} = \sqrt{\det(h)}\; \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$ is the volume form along the fiber $\mathcal{L}$. In this computation, $\delta^{\mu_1\nu_1}_{\rho_1\rho_2}$ is the Kronecker tensor of degrees $r = 2$ which acts on any antisymmetric field $T_{a_1a_2}$ as $\delta^{b_1b_2}_{a_1a_2}T_{b_1b_2} = 2!T_{a_1a_2}$. Using a similar computation, the second term gives:

$$\sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1a_1}, \widehat{F}_{\mu_2a_2})\; \epsilon_{\rho_1\dots\rho_m}\epsilon_{b_1\dots b_n}\; g^{\mu_2\rho_1}h^{a_2b_1}$$

$$\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\rho_2} \wedge \dots \wedge \mathrm{d}x^{\rho_m} \wedge \mathring{\omega}^{b_2}_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^{b_n}_{\mathrm{loc}}$$

$$= \sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1a_1}, \widehat{F}_{\mu_2a_2})\; \epsilon_{\rho_1\dots\rho_m}\epsilon_{b_1\dots b_n}\; g^{\mu_2\rho_1}h^{a_2b_1}$$

$$\tfrac{1}{m!n!}\epsilon^{\mu_1\rho_2\dots\rho_m}\epsilon^{a_1b_2\dots b_n}\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1a_1}, \widehat{F}_{\mu_2a_2})\; \delta^{\mu_1}_{\rho_1}\; \delta^{a_1}_{b_1}\; g^{\mu_2\rho_1}h^{a_2b_1}\; \tfrac{(m-1)!(n-1)!}{m!n!}$$

$$\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \tfrac{(m-1)!(n-1)!}{m!n!}\; \sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{\mu_1a_1}, \widehat{F}_{\mu_2a_2})\; g^{\mu_2\mu_1}h^{a_2a_1}$$

$$\mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^m \wedge \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}$$

$$= \tfrac{1}{mn}\; h(\widehat{F}_{\mu_1a_1}, \widehat{F}_{\mu_2a_2})\; g^{\mu_2\mu_1}h^{a_2a_1}\; \mathrm{dvol} \wedge \omega_{\mathrm{vol}} \tag{7.1.5}$$

For the last term, one obtains:

$$\frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{a_1b_1}, \widehat{F}_{a_2b_2})\; \epsilon_{\rho_1\dots\rho_m}\epsilon_{c_1\dots c_n}\; h^{a_2c_1}h^{b_2c_2}$$

$$\mathrm{d}x^{\rho_1} \wedge \ldots \wedge \mathrm{d}x^{\rho_m} \wedge \mathring{\omega}_{\mathrm{loc}}^{a_1} \wedge \mathring{\omega}_{\mathrm{loc}}^{b_1} \wedge \mathring{\omega}_{\mathrm{loc}}^{c_3} \wedge \ldots \wedge \mathring{\omega}_{\mathrm{loc}}^{c_n}$$

$$= \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{a_1 b_1}, \widehat{F}_{a_2 b_2})\; \epsilon_{\rho_1 \ldots \rho_m} \epsilon_{c_1 \ldots c_n}\; h^{a_2 c_1} h^{b_2 c_2}$$

$$\tfrac{1}{m!n!} \epsilon^{\rho_1 \ldots \rho_m} \epsilon^{a_1 b_1 c_3 \ldots c_n}\; \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \wedge \mathring{\omega}_{\mathrm{loc}}^{1} \wedge \ldots \wedge \mathring{\omega}_{\mathrm{loc}}^{n}$$

$$= \frac{1}{2}\sqrt{\det(h)}\sqrt{\det(g)}\; h(\delta^{a_1 b_1}_{c_1 c_2} \widehat{F}_{a_1 b_1}, \widehat{F}_{a_2 b_2})\; h^{a_2 c_1} h^{b_2 c_2}\; \tfrac{(n-2)!}{n!}\; \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \wedge \mathring{\omega}_{\mathrm{loc}}^{1} \wedge \ldots \wedge \mathring{\omega}_{\mathrm{loc}}^{n}$$

$$= \tfrac{(n-2)!}{n!}\; \sqrt{\det(h)}\sqrt{\det(g)}\; h(\widehat{F}_{c_1 c_2}, \widehat{F}_{a_2 b_2})\; h^{a_2 c_1} h^{b_2 c_2}\; \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \wedge \mathring{\omega}_{\mathrm{loc}}^{1} \wedge \ldots \wedge \mathring{\omega}_{\mathrm{loc}}^{n}$$

$$= \tfrac{1}{n(n-1)}\; h(\widehat{F}_{c_1 c_2}, \widehat{F}_{a_2 b_2})\; h^{a_2 c_1} h^{b_2 c_2}\; \mathrm{dvol} \wedge \omega_{\mathrm{vol}} \qquad (7.1.6)$$

So that, we finally obtain:

$$h(\widehat{F}, \star\widehat{F}) = \Big\{ \frac{1}{m(m-1)}\; h(\widehat{F}_{\rho_1 \rho_2}, \widehat{F}_{\mu_2 \nu_2})\; g^{\mu_2 \rho_1} g^{\nu_2 \rho_2}$$
$$+ \frac{1}{mn}\; h(\widehat{F}_{\mu_1 a_1}, \widehat{F}_{\mu_2 a_2})\; g^{\mu_2 \mu_1} h^{a_2 a_1}$$
$$+ \frac{1}{n(n-1)}\; h(\widehat{F}_{c_1 c_2}, \widehat{F}_{a_2 b_2})\; h^{a_2 c_1} h^{b_2 c_2} \Big\} \mathrm{dvol} \wedge \omega_{\mathrm{vol}}$$

With this expression we can apply the inner integral simply by "reading" the decomposition of $h(\widehat{F}, \star\widehat{F})$. Directly, we see that the Lagrangian density is explicitly written in terms of the components $\widehat{F}_{\mu\nu}$, $\widehat{F}_{\mu a}$ and $\widehat{F}_{ab}$ as

$$\mathcal{L}[\varpi] = \frac{1}{m(m-1)}\; h(\widehat{F}_{\mu_1 \nu_1}, \widehat{F}_{\mu_2 \nu_2})\; g^{\mu_1 \mu_2} g^{\nu_1 \nu_2}$$
$$+ \frac{1}{mn}\; h(\widehat{F}_{\mu_1 a_1}, \widehat{F}_{\mu_2 a_2})\; g^{\mu_1 \mu_2} h^{a_1 a_2}$$
$$+ \frac{1}{n(n-1)}\; h(\widehat{F}_{a_1 a_2}, \widehat{F}_{b_1 b_2})\; h^{a_1 a_2} h^{b_1 b_2} \in C^\infty(\mathcal{M})$$

It is straightforward to check that $\mathcal{L}[\varpi]$ is gauge invariant with respect to the algebraic infinitesimal gauge action of L, as expected by the definition of $\mathcal{S}_{Gauge}[\varpi]$. In the case of the Atiyah Lie algebroid, this gauge invariance is also obtained with respect to the action of the gauge group $\mathcal{G}$. Actually, we can check that each of the three terms $h(\widehat{F}_{\rho_1 \rho_2}, \widehat{F}_{\mu_2 \nu_2})$, $h(\widehat{F}_{\mu_1 a_1}, \widehat{F}_{\mu_2 a_2})$ and $h(\widehat{F}_{c_1 c_2}, \widehat{F}_{a_2 b_2})$ are separately gauge invariant quantities.

In section 4.2.3, we have seen that the decomposition of $\widehat{F}$ in the mixed local basis leads to the homogeneous gluing relations of the three components $\widehat{F}_{\mu\nu}$, $\widehat{F}_{\mu a}$ and $\widehat{F}_{ab}$ with respect to changes of trivializations. These gluing relations are of the forms $\widehat{F}_{j\,\mu\nu} = \alpha^j_i(\widehat{F}_{i,\,\mu\nu})$, $\widehat{F}_{j,\,\mu a} = \widehat{\alpha}^j_i(\widehat{F}_{i,\,\mu a})$ and $\widehat{F}_{j,\,ab} = \widehat{\alpha}^j_i(\widehat{F}_{i,\,ab})$, where $\widehat{F}_{i,\,\mu\nu}$, $\widehat{F}_{i,\,\mu a}$ and $\widehat{F}_{i,\,ab}$ are elements of the space $C^\infty(\mathcal{U}_{ij}) \otimes \mathfrak{g}$. The gluing functions $\alpha^i_j : \Gamma(\mathcal{U}_{ij} \times \mathfrak{g}) \to \Gamma(\mathcal{U}_{ij} \times \mathfrak{g})$ are represented on the basis of $\mathfrak{g}$ by the matrix-valued functions $(G_{ij})^b_a$. These terms are exactly compensated by the gluing transformations of the local expressions h^{ab} of the inner metric h. Then, the Lagrangian density is globally defined over $\mathcal{M}$, as it is expected by definition of $\mathcal{S}_{gauge}[\varpi]$.

7.1.3 Yang-Mills-Higgs model

In this section, we express the Lagrangian density in terms of the local gauge fields A_μ, $\mathring{A}_\mu$ and τ_a^b associated with the local trivialization of the induced ordinary connection ω, $\mathring{\omega}$ and the reduced kernel endomorphism τ associated to a generalized connect ϖ, respectively. We use the formulas (4.2.22), (4.2.23) and (4.2.24) to obtain the following expression of the Lagrangian density $\mathcal{L}[A,\tau]$:

$$\begin{aligned}\mathcal{L}[A,\tau] = {} & \lambda_1\, g^{\mu_1\mu_2} g^{\nu_1\nu_2}\, h_{a_1a_2} \\ & \left(\partial_{\mu_1}A^{a_1}_{\nu_1} - \partial_{\nu_1}A^{a_1}_{\mu_1} + A^{b_1}_{\mu_1}A^{c_1}_{\nu_1}C^{a_1}_{b_1c_1} - \tau^{a_1}_{b_1}\left(\partial_{\mu_1}\mathring{A}^{b_1}_{\nu_1} - \partial_{\nu_1}\mathring{A}^{b_1}_{\mu_1} + \mathring{A}^{d_1}_{\mu_1}\mathring{A}^{e_1}_{\nu_1}C^{b_1}_{d_1e_1}\right)\right)\cdot \\ & \left(\partial_{\mu_2}A^{a_2}_{\nu_2} - \partial_{\nu_2}A^{a_2}_{\mu_2} + A^{b}_{\mu_2}A^{c}_{\nu_2}C^{a_2}_{bc} - \tau^{a_2}_{b_2}\left(\partial_{\mu_2}\mathring{A}^{b_2}_{\nu_2} - \partial_{\nu_2}\mathring{A}^{b_2}_{\mu_2} + \mathring{A}^{d_1}_{\mu_2}\mathring{A}^{e_1}_{\nu_2}C^{b_2}_{d_1e_1}\right)\right) \\ & + \lambda_2\, g^{\mu_2\mu_1} h^{a_2a_1} h_{b_1,b_2} \\ & \left(\partial_{\mu_1}\tau^{b_1}_{a_1} + A^{c_1}_{\mu_1}\tau^{d_1}_{a_1}C^{b_1}_{c_1d_1} - \mathring{A}^{c_1}_{\mu_1}\tau^{b_1}_{d_1}C^{d_1}_{c_1a_1}\right)\cdot\left(\partial_{\mu_2}\tau^{b_2}_{a_2} + A^{c_2}_{\mu_2}\tau^{d_2}_{a_2}C^{b_2}_{c_2d_2} - \mathring{A}^{c_2}_{\mu_2}\tau^{b_2}_{d_2}C^{d_2}_{c_2a_2}\right) \\ & + \lambda_3 h_{c_1c_2}h^{a_1a_2}h^{b_1b_2}\ \left(\tau^{c_1}_{d_1}C^{d_1}_{a_1b_1} - \tau^{d_1}_{a_1}\tau^{e_1}_{b_1}C^{c_1}_{d_1e_1}\right)\cdot\left(\tau^{c_2}_{d_2}C^{d_2}_{a_2b_2} - \tau^{d_2}_{a_2}\tau^{e_2}_{b_2}C^{c_2}_{d_2e_2}\right)\end{aligned} \tag{7.1.7}$$

where $\lambda_1 = \frac{1}{4m(m-1)}$, $\lambda_2 = \frac{1}{mn}$ and $\lambda_3 = \frac{1}{4n(n-1)}$ are combinatorial coefficients coming from the definition of the Hodge-star operator. The elements $C^c_{ab} \in \mathbb{R}$ are the constant structures of the Lie algebra $\mathfrak{g}$.

Under this form, the Lagrangian $\mathcal{L}[A,\tau]$ is written in terms of explicit local gauge fields defined on $\mathcal{U}$. It describes massless vector bosons A_μ coupled to a multi-index scalar field τ_a^b embedded into a quartic potential. At this level of description, we can forget about the upper structure used to construct this theory. Here, the background connection $\mathring{A}_\mu$ is not a dynamical field. Background-dependent gauge field theories are detailed [PS95]. This gauge field theory is equipped with a few free parameters which are the metric $g_{\mu\nu}$, the metric h_{ab} and the $\mathcal{L}$.[1] Note that these parameters are present in the three sectors of L$[A,\tau]$. Finally, this theory does describe a *Yang-Mills-Higgs type theory*.

In expression (7.1.7), the three terms of the Lagrangian $\mathcal{L}[A,\tau]$ are associated to the three terms $\rho^*\widehat{R}$, $\widehat{\mathcal{D}}\tau$ and $\widehat{R}_\tau$ of the curvature $\widehat{F}$. The first term $(\rho^*\widehat{R})_{\mu\nu}(\rho^*\widehat{R})^{\mu\nu}$ corresponds to the field strength of the gauge bosons A^a_μ which are given by the geometric component associated to the induced ordinary connection associated to ϖ. The second term $(\widehat{\mathcal{D}}_\phi)_\mu\tau\ (\widehat{\mathcal{D}}_\phi)^\mu\tau$ corresponds to the covariant derivative of the scalar field used in the mechanism of spontaneous symmetry breaking and gives a minimal coupling between the scalars fields τ_a^b and the gauge bosons A_μ. This term goes with a background connection term depending on $\mathring{A}$ and τ. The last term $\widehat{R}_\tau\widehat{R}_\tau$ is a potential term for the scalar fields τ_a^b. This potential is quartic in τ and is strictly analog to the potential term which embeds the scalar field of the BEHHGK mechanism. This term is positive and it is minimized for $\widehat{R}_\tau = 0$. Here, this minimum has an algebraic interpretation : it corresponds to choose τ as an endomorphism of Lie algebras over L.

7.1.4 Minimal coupling with matter fields

Adding a representation space of A, the previous YMH theory is extended by a coupling term with matter fields. Here, we consider only the case of spineless matter fields s. The

[1] For physical reasons, I do not consider the dimension of $\mathcal{M}$ as a free parameter.

action functional associated to this minimal coupling is defined by using the generalized covariant derivative associate to ϖ. This action functional $\mathcal{S}_{Matter}[\varpi, s]$ is constructed in the same way as $\mathcal{S}_{Gauge}[\varpi]$. It results in two terms: the first one gives the minimal coupling between s and the gauge bosons A_μ and the second one gives a coupling of s with thescalar fields τ_a^b.

For spinor fields, a representation of a Clifford algebra has to be defined on a Dirac bundle. The mathematical formalism of spinors, from the point of view of the geometry, is detailed in [LM89]. In the context of transitive Lie algebroids, this Clifford algebra should be defined by taking into account both $\Gamma(T\mathcal{M})$ and the kernel L. This construction is out of the scope of the present PhD thesis. The generalization of Dirac bundles to transitive Lie algebroids is under investigation.

Let $\mathcal{E}$ be a representation space for the Lie algebroid A equipped with $\phi : \mathsf{A} \to \mathfrak{D}(\mathcal{E})$. Using the generalized covariant derivative defined in section 4.2.7, we construct a gauge invariant term associated to the section $s \in \Gamma(\mathcal{E})$ and a generalized connection ϖ. This term is interpreted as a minimal coupling of the gauge bosons and the field τ with matter. The covariant derivative $\widehat{\mathcal{D}}_\phi$ associated to a generalized connection on A is a 1-form defined on A with values in the first-order derivatives of $\Gamma(\mathcal{E})$. From the relation (4.2.10), this covariant derivative can be written in the mixed local basis as

$$(\widehat{\mathcal{D}}_\phi)_{\mathrm{loc}} = (\widehat{\mathcal{D}}_\phi)_\mu \mathrm{d}x^\mu + (\widehat{\mathcal{D}}_\phi)_a \mathring{\omega}^a_{\mathrm{loc}} \tag{7.1.8}$$

where $(\widehat{\mathcal{D}}_\phi)_\mu$ and $(\widehat{\mathcal{D}}_\phi)_a$ are derivations of $\Gamma(\mathcal{E})$ for $\mu = 1, \dots, m$ and $a = 1, \dots, n$ written as $(\widehat{\mathcal{D}}_\phi)_\mu = \partial_\mu + \phi_{\mathsf{L},\mathrm{loc}}(A_\mu)$ and $(\widehat{\mathcal{D}}_\phi)_a = \phi_{\mathsf{L},\mathrm{loc}}(\tau_a^b E_b)$.

This decomposition of $\widehat{\mathcal{D}}_\phi$ in the mixed local basis indicates that the Hodge star operator can be defined on the covariant derivative in order to obtain $\star\widehat{\mathcal{D}}_\phi$. It results in a $(m+n-1)$-form defined on A with values in the first order derivatives of $\Gamma(\mathcal{E})$. From this, we define the 'action functional associated to the matter field s as

$$\mathcal{S}_{Matter}[\varpi, s] = \int_{\mathsf{A}} h_{\mathcal{E}}(\widehat{\mathcal{D}}_\phi s, \star\widehat{\mathcal{D}}_\phi s), \tag{7.1.9}$$

where $h_{\mathcal{E}}$ describes a ϕ_{L}-compatible metric on $\mathcal{E}$. For the same reasons as in 7.1.1, the Hodge star product commutes with the algebraic infinitesimal action of L (or $\mathcal{G}$, if it exists) and then we straightforwardly obtain $h_{\mathcal{E}}(\widehat{\mathcal{D}}_\phi^\xi s^\xi, \star\widehat{\mathcal{D}}_\phi^\xi s^\xi) = h_{\mathcal{E}}(\widehat{\mathcal{D}}_\phi s, \star\widehat{\mathcal{D}}_\phi s)$, for any $\xi \in \mathsf{L}$ and $s \in \Gamma(\mathcal{E})$. Then, $\mathcal{S}_{Matter}[\varpi, s]$ is gauge invariant with respect to the infinitesimal algebraic action of L. The same result is obtained using the algebraic action of $\mathcal{G}$. Using the definition of the Hodge-star operator, we compute explicitly the quantity $h_{\mathcal{E}}(\widehat{\mathcal{D}}_\phi s, \star\widehat{\mathcal{D}}_\phi s)$ and we obtain

$$\begin{aligned}
h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)s, \star(\widehat{\mathcal{D}}_\phi)s) &= \sqrt{\det(h)}\sqrt{\det(g)}\; h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_\mu s, (\widehat{\mathcal{D}}_\phi)_\nu s)\; \epsilon_{\nu_1\dots\nu_m}\epsilon_{a_1\dots a_n}\; g^{\nu\nu_1} \\
&\qquad \mathrm{d}x^\mu \wedge \mathrm{d}x^{\nu_2} \wedge \dots \wedge \mathrm{d}x^{\nu_m} \otimes \mathring{\omega}^{a_1}_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^{a_n}_{\mathrm{loc}} \\
&\quad + \sqrt{\det(h)}\sqrt{\det(g)}\; h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_a s, (\widehat{\mathcal{D}}_\phi)_b s)\; \epsilon_{\nu_1\dots\nu_m}\epsilon_{a_1\dots a_n}\; h^{ba_1} \\
&\qquad \mathrm{d}x^{\nu_1} \wedge \dots \wedge \mathrm{d}x^{\nu_m} \otimes \mathring{\omega}^{a}_{\mathrm{loc}} \wedge \mathring{\omega}^{a_2}_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^{a_n}_{\mathrm{loc}} \\
&= \sqrt{\det(h)}\sqrt{\det(g)}\; h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_\mu s, (\widehat{\mathcal{D}}_\phi)_\nu s)\; g^{\nu\nu_1} \\
&\quad \frac{(m-1)!}{m!}\delta^\mu_{\nu_1}\; \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \dots \wedge \mathrm{d}x^m \otimes \mathring{\omega}^1_{\mathrm{loc}} \wedge \dots \wedge \mathring{\omega}^n_{\mathrm{loc}}
\end{aligned}$$

$$
+\sqrt{\det(h)}\sqrt{\det(g)}\ h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_a s, (\widehat{\mathcal{D}}_\phi)_b s)\ h^{ba_1}
$$

$$
\frac{(n-1)!}{n!}\delta^a_{a_1}\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_m}\otimes\mathring{\omega}^a_{\mathrm{loc}}\wedge\mathring{\omega}^{a_2}_{\mathrm{loc}}\wedge\ldots\wedge\mathring{\omega}^{a_n}_{\mathrm{loc}}
$$

$$
=\left(\frac{1}{m}\ h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_\mu s, (\widehat{\mathcal{D}}_\phi)_\nu s)\ g^{\nu\mu}+\frac{1}{n}\ h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_a s, (\widehat{\mathcal{D}}_\phi)_b s)\ h^{ba}\right)\mathrm{dvol}\otimes\omega_{\mathrm{vol}} \quad (7.1.10)
$$

The Lagrangian density $\mathcal{L}_{Matter}[\varpi, s]$ is defined from $\mathcal{S}_{Matter}[\varpi, s]$ as $\mathcal{S}_{Matter}[\varpi, s] = \int_{\mathcal{M}}\mathcal{L}_{Matter}[\varpi, s]\mathrm{dvol}$, where dvol is the volume form on $\mathcal{M}$ so that we obtain the expression $\mathcal{L}_{Matter}[\varpi, s]\mathrm{dvol} = \int_{\mathrm{inner}} h_{\mathcal{E}}(\widehat{\mathcal{D}}_\phi s, \star\widehat{\mathcal{D}}_\phi s)\in C^\infty(\mathcal{M})$. We identify this expression with the result of $h_{\mathcal{E}}\left((\widehat{\mathcal{D}}_\phi)s, \star(\widehat{\mathcal{D}}_\phi)s\right)$ so that we can write

$$
\mathcal{L}_{Matter}[\varpi, s] = \frac{1}{m}\ g^{\mu\nu}\ h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_\mu s, (\widehat{\mathcal{D}}_\phi)_\nu s)\ +\frac{1}{n}\ h^{ab}\ h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_a s, (\widehat{\mathcal{D}}_\phi)_b s) \quad (7.1.11)
$$

We implement the expression of the covariant derivatives $(\widehat{\mathcal{D}}_\phi)_\mu$ and $(\widehat{\mathcal{D}}_\phi)_a$ in this Lagrangian so that we obtain

$$
\begin{aligned}
\mathcal{L}_{Matter}[A,\tau,s] &= \lambda_4\ g^{\mu\nu}\ h_{\mathcal{E}}\left(\partial_\mu s+\phi_{\mathsf{L},\mathrm{loc}}(A_\mu)s, \partial_\nu s+\phi_{\mathsf{L},\mathrm{loc}}(A_\nu)s\right)\\
&\quad + \lambda_5\ h^{ab}\ h_{\mathcal{E}}\left(\phi_{\mathsf{L},\mathrm{loc}}(\tau^c_a E_c)s, \phi_{\mathsf{L},\mathrm{loc}}(\tau^d_b E_d)s\right), \qquad (7.1.12)
\end{aligned}
$$

where $\lambda_4 = \frac{1}{m}$ and $\lambda_5 = \frac{1}{n}$. Directly, we can check that this Lagrangian is invariant with respect the the algebraic action of L. As for $\mathcal{L}[A,\tau]$, this Lagrangian density is also invariant with respect to changes of trivialization of A.

The Lagrangian density associated to this minimal coupling term with a matter field s is divided into two terms. The first one corresponds to the usual minimal coupling between the gauge bosons A_μ and the matter field s. The second one gives a coupling between the scalar field τ^b_a and s. In quantum field theory (QFT), this last term is interpreted as an interaction term.

For Atiyah Lie algebroid associated to the principal fiber bundle $\mathcal{P}(\mathcal{M}, G)$, the representation space is the associated vector bundle $\mathcal{E} = \mathcal{P}\times_\ell F$ with F a vector space and ℓ the left representation of G on F. Then, $s\in\Gamma(\mathcal{E})$ is identified with a G-equivariant map $\mathcal{P}\to F$ and its local description by the pull-back by sections as in sub-section 1.2.3. the representation of the Atiyah Lie algebroid on $\mathcal{E}$ is given by the pull-back of $\Gamma_G(\mathcal{P})$ so that the Lagrangian density (7.1.12) becomes:

$$
\begin{aligned}
\mathcal{L}_{Matter}[A,\tau,s] &= \lambda_4\ g^{\mu\nu}\ h_{\mathcal{E}}\left(\partial_\mu s+\ell_*(A_\mu)s, \partial_\nu s+\ell_*(A_\nu)s\right)\\
&\quad + \lambda_5\ h^{ab}\ h_{\mathcal{E}}\left(\tau^c_a\ell_*(E_c)s, \tau^d_b\ell_*(E_d)s\right) \qquad (7.1.13)
\end{aligned}
$$

Here, the requirement for $h_{\mathcal{E}}$ to be ϕ_{L}-compatible is substituted by the requirement for $h_{\mathcal{E}}$ to be G-invariant *i.e.* $h_{\mathcal{E}}(\ell(g^{-1})s_1, \ell(g^{-1})s_2) = h_{\mathcal{E}}(s_1, s_2)$, for any $s_1, s_2\in\Gamma(\mathcal{E})$ and $g\in G$. Then, with respect to this metric, gluing transformations by change of trivializations are related to the gluing functions $g_{ij}:\mathcal{U}_{ij}\to G$ so that we obtain $h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_{i,\mu}s, (\widehat{\mathcal{D}}_\phi)_{i,\mu}s) = h_{\mathcal{E}}((\widehat{\mathcal{D}}_\phi)_{j,\mu}s, (\widehat{\mathcal{D}}_\phi)_{j,\mu}s)$, for any $s:\mathcal{U}_{ij}\to\mathcal{E}$. Same thing for the covariant derivative $(\widehat{\mathcal{D}}_\phi)_{i,a}$.

7.2 Applications of the gauge field theory model

The gauge invariant functional action $\mathcal{S}_{Gauge}[\varpi]$ is defined in the general framework of transitive Lie algebroids. In this section, we give explicit applications of this model in the case of Atiyah Lie algebroids associated to fiber bundle with structure group $U(1)$ and $SU(N)$. These particular cases give results closely related to the gauge theories of the SM.

7.2.1 Application to a $U(1)$-gauge theory

In the SM, $U(1)$-gauge theories are related to the description of electromagnetism. This example permits to see that gauge field theories based on transitive Lie algebroids are compatible with the gauge action of abelian groups.

Consider an Atiyah Lie algebroid associated to the principal bundle $\mathcal{P}(\mathcal{M}, U(1))$, where $\mathcal{M}$ denotes the spacetime ($m = 4$) and $U(1)$ denotes the group of unitary matrix of rank 1 ($n = 1$), with $\mathfrak{u}(1) = \mathrm{i}\mathbb{R}$ its Lie algebra.

For $U(1)$-gauge theory, the induced ordinary connection ω is locally written in terms of a local 1-form A with values in $\mathrm{i}\mathbb{R}$, so that the curvature associated to this connection is simply written as $F = \mathrm{d}A$. Also, the local expression of the reduced kernel endomorphism τ becomes an element of $C^\infty(\mathcal{U})$. For this gauge group, matter fields are described by sections of the associated bundle $\mathcal{E} = (\mathcal{P} \times \mathbb{C})/U(1)$. With respect to the gauge transformations given by the gauge group $\underline{U(1)} = \{z : \mathcal{M} \to U(1)\}$, the three fields A_μ, τ and s transform as $A^z_\mu = A_\mu + z^{-1}\partial_\mu z$, $\tau^z = \tau$ and $s^z = z^{-1}s$, for any $z \in \underline{U(1)}$, respectively. Note that, in the abelian case, τ is gauge invariant. Here, the gauge boson A_μ corresponds to the photon of the particle physics.

In order to explicitly write the total Lagrangian $\mathcal{L}^{U(1)}_{gauge}[A, s, \tau]$, we take $\eta = (+ - - -)$ the Minkowski metric and the metric h, is defined as $h(s_1, s_2) = \overline{s_1}{\cdot}s_2$ for any $s_1, s_2 : \mathcal{U} \to \mathbb{C}$, where the bar denotes the complex conjugate. For $U(1)$, the constant structures C^c_{ab} associated to its Lie algebra vanish. Then, we obtain the Lagrangian

$$\begin{aligned}\mathcal{L}^{U(1)}_{gauge}[A, \tau] =& \frac{1}{48}\, h\left(\partial_\mu A_\nu - \partial_\nu A_\mu - \tau{\cdot}\left(\partial_\mu \mathring{A}_\nu - \partial_\nu \mathring{A}_\mu\right), \partial^\mu A^\nu - \partial^\nu A^\mu - \tau{\cdot}\left(\partial^\mu \mathring{A}^\nu - \partial^\nu \mathring{A}^\mu\right)\right)\\ &+ \frac{1}{4}\, h\left(\partial_\mu\tau, \partial^\mu\tau\right) + \frac{1}{4}\, h\left((\mathcal{D}_A)_\mu s, (\mathcal{D}_A)^\mu s\right) + h\left(\tau{\cdot}s, \tau{\cdot}s\right),\end{aligned}$$

where $(\mathcal{D}_A)_\mu s = \partial_\mu + \ell_*(A_\mu)s$ is the covariant derivative of the field s which gives the minimal coupling term between photons and matter fields. This theory describes the propagations of photons A_μ and scalar fields τ interacting with a matter field s. In this case, the field τ is a massless gauge field.

In particle physics, the Higgs field of the SM does not interact with photons. In our model such coupling terms would be, symbolically, of the form $AA\tau\tau CC$, where the A's denote the photon, the τ's denote the scalar field and the C's denote the constant structures of the gauge group. For the case $U(1)$, these constant structures vanish. Thus, coupling terms between the scalar field τ and the photon A automatically give zero.

7.2.2 Application to a $SU(N)$-gauge theory

Consider an Atiyah Lie algebroid associated to the principal bundle $\mathcal{P}(\mathcal{M}, SU(N))$, where $\mathcal{M}$ denotes the spacetime ($m = 4$) and $SU(N)$ denotes the set of unitary matrix with determinant equals to 1 ($n = N^2 - 1$) with $\mathfrak{su}(N)$ its Lie algebra. For the metric g on $\mathcal{M}$, we take the Minkowski metric with the signature $(+---)$, which is denoted by $\eta_{\mu\nu}$, and for the Killing inner metric h, we take $h(\gamma, \gamma') = \mathrm{tr}(\gamma\cdot\gamma')$, where tr denotes the trace on the matrix algebra for any $\gamma, \gamma' \in \mathfrak{su}(N)$.

We use a canonical basis for $SU(N)$ *i.e.* a set $(E_1, \dots, E_n)$ of $N\times N$ matrices such that one has $[E_i, E_j] = C_{ij}^k E_k$, where $C_{ij}^k \in \mathbb{R}$ are the structure constants and $\mathrm{tr}(E_i E_j) = -\delta_{ij}$ for $i, j, k = 1 \dots n$ where δ_{ij} denotes the Kronecker symbol. The elements $(E_i)_{i=1\dots n}$ of the basis of $\mathfrak{su}(N)$ are anti-hermitian matrices *i.e.* $E_i^\dagger = -E_i$, where † denotes the transposed complex-conjugate. Thus, we obtains $h_{ij} = -\delta_{ij}$ and $h^{ij} = -\delta^{ij}$. An element of $\mathfrak{su}(N)$ is written as $\gamma = \gamma^a E_a$ where $\gamma_a \in \mathbb{R}$. For example, for $N = 2$, such a basis is given by $E_1 = \frac{\mathrm{i}}{\sqrt{2}}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $E_2 = \frac{\mathrm{i}}{\sqrt{2}}\left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $E_3 = \frac{\mathrm{i}}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

To describe the matter fields of the theory, we consider the associated vector bundle $\mathcal{E} = \mathcal{P} \times_\ell \mathbb{C}^N$ whose sections are given by the local maps $s : \mathcal{U} \to \mathbb{C}^N$. We also assume that the metric $h_\mathcal{E}$ is defined as $h_\mathcal{E}(s_1, s_2) = \int_\mathcal{M} s_1(p)^\dagger \cdot s_2(p)\, \mathrm{d}x^1 \dots \mathrm{d}x^m$ for any $s_1, s_2 \in \Gamma(\mathcal{E})$, where the † denotes the transposed complex conjugate.

To simplify the computations, we take the geometric component of the background connection $\mathring{A}_\mu = 0$ for $\mu = 1, 2, 3, 4$. [2] The total Lagrangian associated to the $SU(N)$-gauge theory and to minimal coupling with a matter field s becomes:

$$\begin{aligned}
\mathcal{L}^{SU(N)}_{Gauge}[A, s, \tau] = &-\frac{1}{48}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{b_1c_1} A^{b_1}_\mu A^{c_1}_\nu\right)\left(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a} + C^a_{b_2c_2} A^{\mu\, b_2} A^{\nu\, c_2}\right) \\
&+ \frac{1}{4n}\,(\partial_\mu \tau^b_a + A^{c_1}_\mu \tau^{d_1}_a C^b_{c_1d_1})(\partial^\mu \tau^b_a + A^{\mu\, c_2}\tau^{d_2}_a C^b_{c_2d_2}) \\
&- \frac{1}{4\, n(n-1)}\left(\tau^{d_1}_a \tau^{e_1}_b C^c_{d_1e_1} - C^{d_1}_{ab}\tau^c_{d_1}\right)\left(\tau^{d_2}_a \tau^{e_2}_b C^c_{d_2e_2} - C^{d_2}_{ab}\tau^c_{d_2}\right) \\
&+ \frac{1}{4}\,(\partial_\mu s + \ell_*(A_\mu)s)^\dagger \cdot (\partial^\mu s + \ell_*(A^\mu)s) \\
&- \frac{1}{n}\,(\tau^c_a\, \ell_*(E_c)s)^\dagger \cdot \left(\tau^d_a\, \ell_*(E_d)s\right)
\end{aligned}$$

One recalls the infinitesimal action of the space $\Gamma_G(\mathcal{P}, \mathfrak{g})$ on the fields of the theory. Let $\xi : \mathcal{U} \to \mathfrak{g}$ be the gauge parameter (depending of the point) of the theory. The gauge fields of the theory transform as

$$\begin{cases}
A^a_\mu \mapsto A^a_\mu + \delta A^a_\mu & \text{with} \quad \delta A^a_\mu = A^b_\mu \xi^c C^a_{bc} + \partial_\mu \xi^a \\
\tau^a_b \mapsto \tau^a_b + \delta\tau^a_b & \text{with} \quad \delta\tau^a_b = \tau^c_a \xi^d C^b_{cd} \\
s \mapsto s + \delta s & \text{with} \quad \delta s = -\xi^a \ell_*(E_a)s
\end{cases} \tag{7.2.1}$$

By a direct computation, we check that $\mathcal{L}^{SU(N)}_{Gauge}[A, s, \tau]$ is gauge invariant with respect to the algebraic action of $\Gamma(\mathcal{U} \times \mathfrak{g})$. We can also check that the theory is invariant with respect to a change of trivializations of $\Gamma_G(\mathcal{P})$.

In this $SU(N)$-gauge theory, the mass term associated to the scalar fields τ^b_a are explicitly given in terms of the structure constants of $\mathfrak{su}(N)$. This mass term becomes

$$\delta\mathcal{L}[A, \tau] = \frac{1}{4n(n-1)} C^d_{ab} C^e_{ab} \tau^c_d \tau^c_e \tag{7.2.2}$$

where there is an implicit summation over the repetitive indeces.

[2] Such a background connection can only be defined locally.

7.3 Spaces of solutions

In this section, we give examples of spaces of *solutions* in $\Omega^1(\mathsf{A}, \mathsf{L})$, with respect to the potential term of the theory. Two solutions are given, the first one gives the usual YM model and the second one gives massive vector bosons. These two solutions are related to the "phase transition" which occurs in the spontaneous symmetry breaking mechanism.

7.3.1 Two canonicals examples of $\Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$

In the usual BEHHGK mechanism, the vacuum state for the scalar field ϕ corresponds to its value which minimizes the potential. Depending on the energy of the system, this potential takes either the shape of a *well* or the shape of a *sombrero*, so that the vacuum state of the scalar field ϕ corresponds either to $\phi = 0$ or $\phi \neq 0$, respectively. In YMH type theories based on generalized connections, the potential is not fixed by any dynamical parameters. It is written as an algebraic constraint on τ which is minimal if and only if τ is a morphism of Lie algebras. Then, the so-called vacuum state of the scalar field ϕ would correspond to some specific subspaces of generalized connections. One denotes by $\Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$ the space of generalized connections with their associated reduced kernel endomorphism τ preserves the Lie bracket on L. Directly, we can check that the algebraic action of L (or $\mathcal{G}$ for Atiyah Lie algebroid) is compatible with this subspace in the sense that if $\varpi \in \Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$, then so is ϖ^ξ (or ϖ^g).

The first example of $\Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$ is given by the space of ordinary connections on A, *i.e.* for $\tau = 0$, which is a trivial morphism of Lie algebras on L. According to this value, the Lagrangian is greatly simplified and becomes

$$\begin{aligned}\mathcal{L}[A, s, \tau = 0] = \lambda_1 \, h_{a_1 a_2} \Big(\partial_\mu A^{a_1}_\nu - \partial_\nu A^{a_1}_\mu + A^{b_1}_\mu A^{c_1}_\nu C^{a_1}_{b_1 c_1}\Big) \cdot \\ \Big(\partial^\mu A^{\nu\, a_2} - \partial^\nu A^{\mu\, a_2} + A^b_{\mu_2} A^c_{\nu_2} C^{a_2}_{bc}\Big) \\ + \lambda_4 \, h_{\mathcal{E}} \left(\partial_\mu s + \phi_{\mathsf{L},\text{loc}}(A_\mu) s, \partial^\mu s + \phi_{\mathsf{L},\text{loc}}(A^\mu) s\right)\end{aligned} \tag{7.3.1}$$

which is the standard formulation of the YM model in QFT coupled with a matter field s. This is not surprising to see that YM theories are coming from gauge invariant action functionals based on generalized connections. Indeed, we have seen that YM theories are based on the existence of Ehersmann connections on principal bundles which are in 1:1 correspondence with ordinary connections defined on transitive Lie algebroids. However, it is more surprising to see the YM theories are *a solution* in the more general context of YMH theories based on generalized connections. Comparing with the usual interpretation of the BEHHGK mechanism of spontaneous symmetry breaking, this space of solutions would correspond to the vacuum configuration associated to a well potential.

The solution $\tau = 0$ shows that the well-known YM theories associated to ordinary connections are compatible with the more general framework of the YMH theories associated to generalized connections on transitive Lie algebroids with respect to, at least, two points. Firstly, the space of ordinary connections is a subspace of the space of the generalized connections on A, compatible with the algebraic gauge action of L (or $\mathcal{G}$) which is canonically identified simply by putting $\tau = 0$. Secondly, the YM theories are described as a solution space of our constructions corresponding to this vacuum configuration $\phi = 0$ in the BEHHGK mechanism.

The second example of $\Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$ is given by the space of generalized connections on A such that $\tau(\ell) = \ell$, for any $\ell \in \mathsf{L}$. Indeed, $\tau = \text{Id}_{\mathsf{L}}$ is a morphism of Lie algberas over L.

Then, the Lagrangian can be written as:

$$\begin{aligned}\mathcal{L}[A, \tau = \mathrm{Id}_{\mathsf{L}}] = \lambda_1\ g^{\mu_1\mu_2} g^{\nu_1\nu_2}\ h_{a_1a_2}& \\ \Big(\partial_{\mu_1} A^{a_1}_{\nu_1} - \partial_{\nu_1} A^{a_1}_{\mu_1} + A^{b_1}_{\mu_1} A^{c_1}_{\nu_1} C^{a_1}_{b_1c_1} - \partial_{\mu_1} \mathring{A}^{a_1}_{\nu_1} + \partial_{\nu_1} \mathring{A}^{a_1}_{\mu_1} - \mathring{A}^{d_1}_{\mu_1} \mathring{A}^{e_1}_{\nu_1} C^{a_1}_{d_1e_1}\Big) \cdot& \\ \Big(\partial_{\mu_2} A^{a_2}_{\nu_2} - \partial_{\nu_2} A^{a_2}_{\mu_2} + A^{b}_{\mu_2} A^{c}_{\nu_2} C^{a_2}_{bc} - \partial_{\mu_2} \mathring{A}^{a_2}_{\nu_2} + \partial_{\nu_2} \mathring{A}^{a_2}_{\mu_2} - \mathring{A}^{d_1}_{\mu_2} \mathring{A}^{e_1}_{\nu_2} C^{a_2}_{d_1e_1}\Big)& \\ + \lambda_2\ g^{\mu_2\mu_1} h^{a_2a_1} h_{b_1,b_2}\ \Big(A^{c_1}_{\mu_1} C^{b_1}_{c_1a_1} - \mathring{A}^{c_1}_{\mu_1} C^{b_1}_{c_1a_1}\Big) \cdot \Big(A^{c_2}_{\mu_2} C^{b_2}_{c_2a_2} - \mathring{A}^{c_2}_{\mu_2} C^{b_2}_{c_2a_2}\Big)& \\ + \lambda_4\ g^{\mu\nu} h_{\mathcal{E}}(\partial_\mu s + \phi_{\mathsf{L},\mathrm{loc}}(A_\mu)s, \partial_\nu s + \phi_{\mathsf{L},\mathrm{loc}}(A_\nu)s)& \\ - \lambda_5\ h^{ab} h_{\mathcal{E}}\left(\phi_{\mathsf{L},\mathrm{loc}}(E_a)s, \phi_{\mathsf{L},\mathrm{loc}}(E_b)s\right)& \end{aligned} \tag{7.3.2}$$

For this solution, we see the presence of the mass term $m_{ab}A^a_\mu A^{\mu\, b}$ where m_{ab} denotes the *mass matrix* of the gauge bosons A_μ. Which is written as :

$$m_{ab} = \frac{1}{mn}\ h^{a_1a_2} h_{b_1b_2} C^{b_1}_{aa_1} C^{b_2}_{ba_2} \tag{7.3.3}$$

Contrary to the solution corresponding to $\tau = 0$, this second space of solution gives mass term for the gauge bosons A_μ and thus corresponds to an alternative to the mechanism of spontaneous symmetry breaking. In analogy with this mechanism, this second solution is related to the vacuum configuration of ϕ associated to the potential with a *sombrero* shape which is minimized for non-zero values of ϕ. These two examples show the analogy between the BEHHCK mechanism and the selection of some spaces of solution in $\Omega^1(\mathsf{A}, \mathsf{L})$. Indeed, in both situations, the mass term of the gauge bosons A_μ does come from the interaction with a scalar field. When this scalar field minimizes its potential term, then it results in either a mass term $m = 0$ or $m \neq 0$.

However, there is a major difference between these two mechanisms. Contrary to the usual interpretation of the spontaneous symmetry breaking mechanism, the "shape" of the potential term $R_\tau R_\tau$ is not given in terms of some dynamical parameters related to the energy scale of the system. Here, the potential term is an algebraic constraint so that the "phase transition" is substituted by change of space of solutions, from $\tau = 0$ to $\tau = \mathrm{Id}_{\mathsf{L}}$. For this last choice, the theory describes massive gauge bosons.

Concerning the gauge transformations, the space of solutions associated to $\tau = \mathrm{Id}_{\mathsf{L}}$ is not compatible with the algebraic infinitesimal action of L. Then, the reduced kernel endormiphsm τ would transform as $\tau^\xi(\gamma) = \gamma - [\xi, \gamma]$, for any gauge parameter $\xi \in \mathsf{L}$, so that mass term (7.3.3) is not gauge invariant. Accordingly to the gauge principle, does not correspond to observable in physics. In chapter 8, we give a new method of symmetry reduction of the gauge group. Applied to a specific subspace of $\Omega_{\mathrm{sol}}(\mathsf{A}, \mathsf{L})$ on Atiyah Lie algebroids, this method gives gauge-invariant mass terms for the vectors bosons of the theory.

As an application of this space of solutions, we compute the mass matrix of the gauge bosons A_μ for differential groups of symmetry. To compute the mass matrix, we choose $SU(N)$-gauge theories where the basis of the Lie algebras $\mathfrak{su}(N)$ is given as in section 7.2.2 so that $h_{ab} = -\delta_{ab}$. As examples, one takes $N = 2$, $N = 3$ and $N = 4$. For $N = 2$, one takes $E_1 = \frac{\mathrm{i}}{\sqrt{2}} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $E_2 = \frac{\mathrm{i}}{\sqrt{2}} \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $E_3 = \frac{\mathrm{i}}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The mass matrix becomes:

$$(m_{ab}) = \frac{1}{3}\ \mathrm{Id}_{3\times 3} \tag{7.3.4}$$

For $N = 3$, one takes $E_i = \frac{\mathrm{i}}{\sqrt{2}}\,\lambda_i$, where λ_i denotes the Gell-Mann's matrices. The mass matrix becomes:

$$(m_{ab}) = \frac{3}{6}\,\mathrm{Id}_{8\times 8} \tag{7.3.5}$$

For $N = 4$, the mass matrix becomes:

$$(m_{ab}) = \frac{2}{15}\,\mathrm{Id}_{15\times 15} \tag{7.3.6}$$

The computations have been performed using the Mathematica code presented in the appendix B.

Chapter 8

Method of symmetry reduction

In gauge field theories, one usually have to deal with gauge degrees of freedom in order to create gauge-invariant quantities. In the literature, there exist at least three methods to perform reductions of the symmetry group. Firstly, the spontaneous symmetry breaking mechanism breaking of the SM consists in adding a scalar field in the theory. By a dynamical process, this scalar field operates a polarization and the initial group of symmetry is spontaneously reduced to a residual subgroup (see [PS95; IZ85; Wei95]). Secondly, the gauge-fixing method consists in selecting one representative gauge configuration in a gauge orbit of the theory by adding a constraint equation on a gauge field. In order to preserve this representative element, the gauge group does not act anymore on the fields of the theory (same references as before). Thirdly, the method of reduction of fiber bundle is defined as a mathematical construction on a principal bundle $\mathcal{P}(\mathcal{M}, G)$. This method states that if there exists a section of an associated fiber bundle with fiber isomorph to G/H, where $H \subset G$, then the principal bundle $\mathcal{P}(\mathcal{M}, G)$ is reductive to a H-principal bundle [KN96a].

This chapter is devoted to the construction of an alternative method of symmetry reduction. The general construction is detailed in [FFLM13]. This method is related to the definition of gauge-invariant composite fields out of connections. The general idea of this construction is to make a change of variables in a functional space of gauge fields (ω, Λ), where ω is a gauge connection and Λ is a gauge field defined on $\mathcal{M}$, called the *auxiliary field* associated to this construction. It results in this change of variables that the part of Λ which carries the action of the gauge group is moved to ω in order to define a gauge-invariant composite field $\widehat{\omega}$.

As a first example, we apply this method to the electroweak part $U(1) \times SU(2)$ of the SM of particle physics. The gauge connections are the gauge bosons $a_\mu : \mathcal{M} \to U(1)$ and $b_\mu : \mathcal{M} \to SU(2)$ and the auxiliary field is the scalar field $\phi : \mathcal{M} \to \mathbb{C}^2$. It results in a gauge invariant theory with vector bosons $W^\pm$ and Z^0, interacting with a scalar field $\eta : \mathcal{M} \to \mathbb{R}^+$ defined out of ϕ. *De facto*, this method reduces the symmetry group $U(1) \times SU(2)$ to $U(1)$ without making reference neither to any gauge fixing term nor to any symmetry breaking mechanism. In section 8.2.2, we will see that this construction does work only for $SU(2)$ gauge theories and, for dimension $N > 2$, obstruction terms occur and the "neutralization" of the gauge group $SU(N)$ is no more possible.

We also apply this mechanism in the context of gauge theories based on generalized connections on transitive Lie algebroids. To do so, we consider Atiyah Lie algebroids associated to a principal bundle $\mathcal{P}(\mathcal{M}, SU(N))$ and we restrict the space of generalized connections to the space $\Omega^1_{\text{sol}}(\mathsf{A}, \mathsf{L})$ whose reduced kernel endomorphisms τ can be written as $\tau(\gamma) = u^{-1}\gamma u$, where u is an element of the gauge group.[1] Then, the induced ordinary

[1]The reader should not confuse this "u field" with the point $u \in \mathcal{P}$ of the previous sections. In this section, the points of $\mathcal{P}$ are denoted by the letter w.

connection ω, associated to ϖ ,“absorbs” the u field so that it defines a field $\widehat{\omega}$, which we denote as the gauge-invariant composite field. This procedure works for any dimension N.

8.1 General framework

The method of symmetry reduction depicted in this chapter is based on the existence of a gauge connection with a given representation of the gauge group $\mathcal{G}$ and a group-valued field. This group-valued field is called the *dressing field* of the theory. Practically, this dressing field is defined out of an auxiliary field, not necessarily with values in the gauge group, which is called the *auxiliary field* of the theory. Then, the dressing field would correspond to the "part" of the auxiliary field which supports the representation of $\mathcal{G}$. Then, gauge connections and dressing fields are "mixed" together, in an appropriate manner, so that it results in a gauge-invariant composite field.

We consider a gauge field theory associated to a principal bundle $\mathcal{P}(\mathcal{M}, H)$ with structure group H and its Lie algebra $\mathfrak{h}$. We denote by $\mathcal{H}$ the gauge group associated to H, *i.e.* the space $\{h : \mathcal{P} \to H | h(w{\cdot}a) = a^{-1}h(w)a, \forall w \in \mathcal{P} \text{ and } \forall a \in H\}$. Let ω be a connection 1-form defined on $\mathcal{P}(\mathcal{M}, H)$ which transforms with respect to the gauge group $\mathcal{H}$ as

$$\omega^h = h^{-1}\omega h + h^{-1}\widehat{\mathrm{d}}h \tag{8.1.1}$$

where $h \in \mathcal{H}$ and $\widehat{\mathrm{d}}$ is a convenient graded differential operator.

Let G be a Lie group such that $H \subset G$ and let u be a G-valued field over $\mathcal{M}$ which varies under a gauge transformation as $u^h = h^{-1}u$ for any $h \in \mathcal{H}$. Such a field is called the *dressing field* of the theory. Then, if it exists, we define the *composite field* associated to the connection 1-form ω and the dressing field u as the field

$$\widehat{\omega} := u^{-1}\omega u + u^{-1}\widehat{\mathrm{d}}u. \tag{8.1.2}$$

If it makes sense, this composite field is a gauge-invariant field. Indeed, we consider that the fields u and ω have their own representation of $\mathcal{H}$ so that the gauge transformation of $\widehat{\omega}$ is defined as $\widehat{\omega}^h = (u^h)^{-1}\omega^h u^h + (u^h)^{-1}\widehat{\mathrm{d}}u^h$, for any $h \in \mathcal{H}$. A direct computation shows that

$$\widehat{\omega}^h = u^{-1}{\cdot}hh^{-1}\omega hh^{-1}{\cdot}u + u^{-1}{\cdot}(\widehat{\mathrm{d}}h)h^{-1}{\cdot}u + u^{-1}{\cdot}h(\widehat{\mathrm{d}}h^{-1}){\cdot} + u^{-1}\mathrm{d}u = u^{-1}\omega u + u^{-1}\widehat{\mathrm{d}}u = \widehat{\omega}. \tag{8.1.3}$$

The gauge invariance of $\widehat{\omega}$ occurs *de facto* from an appropriate combination of the u fields and the connection ω. The action of $\mathcal{H}$ is not forbidden but instead, it is “neutralized” in the sense that its representation on $\widehat{\omega}$ becomes trivial. Note that the composite field $\widehat{\omega}$ does not always belong to a space of connections on $\mathcal{P}(\mathcal{M}, H)$, *e.g.* for $G \neq H$.

The method used to construct gauge invariant composite fields can be extended to create gauge-invariant composite fields out of matter fields. Indeed, we apply this procedure to matter fields in order to obtain a gauge-invariant composite field. Let $\mathcal{E} = \mathcal{P} \times_\ell \mathcal{F}$ be a vector bundle associated to the principal bundle $\mathcal{P}(\mathcal{M}, H)$. The gauge group $\mathcal{H}$ acts on $s \in \Gamma(\mathcal{E})$ as $s^h = \ell(h^{-1})s$, for any $h \in \mathcal{H}$ so that the gauge invariant composite field associated to s and the dressing field u is defined as

$$\widehat{s} := \ell(u^{-1})s \tag{8.1.4}$$

For the same reason a before, the gauge transformation of $\widehat{s}$ is given as $\widehat{s}^h = \ell((u^h)^{-1})s^h = \ell(u^{-1}h)\ell(h^{-1})s = \ell(u^{-1})s = \widehat{s}$, for any $h \in \mathcal{H}$. With respect to this invariance, the composite field $\widehat{s}$ is no more a section on the fiber bundle $\mathcal{E}$.

Gauge invariant composite fields can also be defined out of the curvature R associated to a connection ω or the covariant derivative $\mathcal{D}s$ along ω of a section s. It results in gauge invariant composite objects, compatible with their corresponding composite fields $\widehat{\omega}$ and $\widehat{s}$.

We start with the curvature R associated to the connection 1-form ω defined as $R = d\omega + \frac{1}{2}[\omega, \omega]$, where $[\cdot, \cdot]$ denotes the graded Lie bracket adapted to the differential structure of ω. We define the gauge invariant composite field $\widehat{R} := u^{-1}Ru$, which is no more a curvature on $\mathcal{P}$. Moreover, using a straightforward computation, we show that $\widehat{R} = d\widehat{\omega} + \frac{1}{2}[\widehat{\omega}, \widehat{\omega}]$ so that, in appearance, $\widehat{R}$ would be the "curvature" associated to $\widehat{\omega}$.

The gauge-invariant composite covariant derivative can also be defined in terms of the composite connection $\widehat{\omega}$. Let $\mathcal{D}$ be the covariant derivative associated to the connection 1-form ω acting on the field s as $\mathcal{D}s = d_{\mathcal{E}}s + \ell_*(\omega)s$, where d denotes a graded differential operator associated to the representation space s and ℓ_* the induced representation of $\mathfrak{h}$ on $\Gamma(\mathcal{E})$. Then, from (8.1.4), the composite field $\widehat{\mathcal{D}s} = \ell(u^{-1})\mathcal{D}s$ is gauge-invariant and we show directly that $\widehat{\mathcal{D}s} := \widehat{\mathcal{D}}\widehat{s}$, where $\widehat{\mathcal{D}}s := d_{\mathcal{E}}s + \ell_*(\widehat{\omega})s$, for any $s \in \Gamma(\mathcal{E})$.

Consider a gauge theory defined by a Lagrangian $\mathcal{L}[\omega, s, \phi]$ where ω is a connection 1-form, s is a matter field and ϕ is a gauge field which can be decomposed as (u, η) with u is a dressing field and η a gauge invariant field called the *residual field*. By using the dressing u, we define the composite fields $\widehat{\omega}$ and $\widehat{s}$. The gauge invariance of the Lagrangian permits to claim that, under its final form, the theory does not depend explicitly on the u field so that we obtain $\mathcal{L}[\omega, s, \phi] = \mathcal{L}[\widehat{\omega}, \widehat{s}, \eta]$. Written in the new variables, the gauge group does not act anymore on the fields of the theory. This assertion is illustrated by the following examples. A counting of the degrees of freedom of the fields before and after the change of variables shows that this method preserves the degrees of freedom of the theory, as expected.

8.2 Application to $SU(N)$-gauge theories

The electroweak part of the SM describes the propagation of the gauge bosons associated to the group of symmetry $U(1) \times SU(2)$, interacting with an external scalar field $\phi : \mathcal{M} \to \mathbb{C}^2$ embedded into a quartic potential. This scalar field corresponds to an auxiliary field from which we define a dressing field $u : \mathcal{M} \to SU(2)$. Then, we define the gauge invariant composite fields of the theory and we show that they exactly correspond, in an appropriate basis of $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, to the photon A_μ and the vector bosons $W^\pm_\mu$ and Z_μ. All the fields of the theory are invariant with respect to the symmetry $SU(2)$ but not with respect to $U(1)$.

This method is extended to a $SU(N)$-gauge theory coupled with a scalar field $\phi : \mathcal{M} \to \mathbb{C}^N$. We shows that, for $N > 2$, the dressing field associated to ϕ is not unique so that the previous method cannot be rigorously applied.

8.2.1 The electroweak part of the Standard Model

Consider the trivial fiber bundle $\mathcal{P} = \mathcal{M} \times G$, where $G = U(1) \times SU(2)$. We denote by $\underline{U(1)}$ and $\underline{SU(2)}$ the sets of functions $\zeta : \mathcal{M} \to U(1)$ and $\gamma : \mathcal{M} \to SU(2)$, respectively. They form the gauge groups associated to G. The Lagrangian associated to the electroweak part of the SM is given as follows

$$\mathcal{L}[a_\mu, b_\mu, \phi] = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu} - \frac{1}{4} \sum_a g^a_{\mu\nu} g^{\mu\nu\, a} + (\mathcal{D}_\mu \phi)^\dagger (\mathcal{D}_\mu \phi) - \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \tag{8.2.1}$$

where $f_{\mu\nu}$ and $g_{\mu\nu}$ are the field strengths associated to the $U(1)$-connection a_μ and to the $SU(2)$-connection b_μ, the covariant derivative is given as $\mathcal{D}_\mu = \partial_\mu + \frac{2i}{g'} a_\mu + \frac{2i}{g} b_\mu$, the field $\phi : \mathcal{M} \to \mathbb{C}^2$ is the scalar field associated to the BEHHGK mechanism of spontaneous symmetry breaking and the symbol $\dagger$ denotes the transposed complex conjugate,

This theory is invariant with respect to the following gauge transformations. The first line and the second line correspond to the transformations with respect to the gauge groups $\underline{U(1)}$ and $\underline{SU(2)}$, respectively.

$$\begin{array}{llll} \underline{U(1)}: & a^\zeta_\mu = a_\mu + \frac{2i}{g'}\zeta^{-1}\partial_\mu \zeta & b^\zeta_\mu = b_\mu & \phi^\zeta = \zeta^{-1}\phi \\ \underline{SU(2)}: & a^\gamma_\mu = a_\mu & b^\gamma_\mu = \gamma^{-1} b_\mu \gamma + \frac{2i}{g}\gamma^{-1}\partial_\mu\gamma & \phi^\gamma = \gamma^{-1}\phi, \end{array} \tag{8.2.2}$$

where g' and g denote the coupling constants associated to the gauge groups $U(1)$ and $SU(2)$, respectively.

From the scalar field ϕ, we define a field $u : \mathcal{M} \to SU(2)$ which transforms with respect to $\underline{SU(2)}$ as $u^\gamma = \gamma^{-1} u$. To see this, we use the fact that elements of $\mathbb{C}^2 \backslash \{0\}$ are uniquely decomposed as $\mathbb{C}^2\backslash\{0\} = SU(2) \otimes \mathbb{R}^*_+$. This decomposition of ϕ becomes explicit as one chooses a reference vector $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ such that the scalar field $\phi = \left(\begin{smallmatrix} \phi_1 \\ \phi_2 \end{smallmatrix}\right)$ can be decomposed as $\phi = \eta u \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, where $\eta = \sqrt{|\phi_1|^2 + |\phi_2|^2} \in \mathbb{R}^+$ denotes the norm of ϕ and $u : \mathcal{M} \to SU(2)$ is defined as

$$u = (1/\eta) \left(\begin{smallmatrix} \overline{\phi_2} & \phi_1 \\ -\overline{\phi_1} & \phi_2 \end{smallmatrix}\right). \tag{8.2.3}$$

As the norm of ϕ, the field η is gauge invariant with respect to the action of both $\underline{U(1)}$ and $\underline{SU(2)}$. Then, the matrix-valued field u carries all the representation of the gauge group. Directly, we compute the gauge transformation of u with respect to $\gamma \in \underline{SU(2)}$ and $\zeta \in \underline{U(1)}$ and we obtain

$$u^\gamma = \gamma^{-1} u \quad ; \quad u^\zeta = u \cdot \widehat{\zeta}, \tag{8.2.4}$$

where $\widehat{\zeta} = \left(\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right)$. The first transformation shows that the field u corresponds to a dressing field with respect to the gauge group $\underline{SU(2)}$. Then, we define the composite field B_μ associated to the gauge field b_μ and the dressing field u as

$$B_\mu = u^{-1} b_\mu u + u^{-1}\partial_\mu u, \tag{8.2.5}$$

which is gauge invariant with respect to the action of $\underline{SU(2)}$. However, it is not invariant with respect to the action of $\underline{U(1)}$. The gauge field a_μ is not concerned with the u field. Then, in the functional space of fields of the theory, we have conducted the following change of variables:

$$(a_\mu, b^1_\mu, b^2_\mu, b^3_\mu, \phi) \mapsto (a_\mu, B^1_\mu, B^2_\mu, B^3_\mu, u, \eta) \tag{8.2.6}$$

In particle physics, in order to make apparent the vector bosons corresponding to the electroweak interaction, one proceeds to a rotation of the basis of $\mathfrak{u}(1) \oplus \mathfrak{su}(2)$, measured by the so-called *weak angle* θ_W. This rotation leads to the definition of the photon A_μ, the $U(1)$-charged bosons $W^\pm_\mu$ and the neutral boson Z_μ so that we have

$$(a_\mu, B^1_\mu, B^2_\mu, B^3_\mu) \mapsto (A_\mu, W^+_\mu, W^-_\mu, Z_\mu) \tag{8.2.7}$$

where

$$\begin{cases} A_\mu = \sin\theta_W B^3_\mu + \cos\theta_W a_\mu \\ W^\pm_\mu = \frac{1}{\sqrt{2}}(B^1_\mu \mp i B^2_\mu) \\ Z_\mu = \cos\theta_W B^3_\mu - \sin\theta_W a_\mu \end{cases} , \tag{8.2.8}$$

with $\cos\theta_W = \frac{g}{g^2+g'^2}$ and $\sin\theta_W = \frac{g'}{g^2+g'^2}$. Directly, we can check that all these fields are invariant with respect to the action of $\underline{SU(2)}$ and that they does transform with respect to $\underline{U(1)}$ as

$$A^\zeta_\mu = A_\mu + \tfrac{2i}{e}\zeta^{-1}\partial_\mu\zeta \quad ; \quad (W^\pm_\mu)^\zeta = \zeta^{\mp 2} W^\pm_\mu \quad ; \quad Z^\zeta_\mu = Z_\mu. \tag{8.2.9}$$

Then, the gauge group $\underline{U(1)}$ is only represented on the photon A_μ and the bosons $W^\pm_\mu$. These transformations show that A_μ is a local $U(1)$-connection on $\mathcal{P}$ and $W^\pm_\mu$ correspond to the $U(1)$-charged bosons of the SM.

The Lagrangian $\mathcal{L}[a_\mu, b_\mu, \phi]$ can now be written in terms of the new composite fields A_μ, $W^\pm_\mu$, Z^0_μ and the new fields u and η. Using the gauge invariance of the theory, it is straightforward to show that the u fields do not appear in the theory as dynamical fields so that, in the new variables, this theory depends only on the gauge-invariant composite vector bosons and the norm η and we finally have

$$\mathcal{L}[a_\mu, b_\mu, \phi] = \mathcal{L}[a_\mu, B_\mu, \eta] \mapsto \mathcal{L}'[A_\mu, W^\pm_\mu, Z_\mu, \eta] \tag{8.2.10}$$

The Lagrangian $\mathcal{L}'[A_\mu, W^\pm_\mu, Z_\mu, \eta]$ is gauge invariant since all the fields of the theory are invariant with respect to the action of $\underline{SU(2)}$ and then could correspond to observables in Physics. All these fields are not invariant with respect to $\underline{U(1)}$, but the Lagrangian still preserves its gauge-invariance with respect to this gauge group. The action of $\underline{SU(2)}$ has been "neutralized" so that the method described here has permitted the reduction of the gauge-group $\underline{U(1)} \times \underline{SU(2)} \to \underline{U(1)}$.

The residual field $\eta \in \mathbb{R}^+$ is now embedded into the potential $-\phi^2\eta - \lambda\phi^4 = -\mu^2\eta - \lambda\eta^4$ and can be extended as $\eta = \eta_0 + H$, where η_0 minimizes this potential term and H corresponds to the propagation of the Higgs field. The perturbation of η around η_0 permits also the attribution of mass terms associated to the vector bosons $W^+_\pm$ and Z_μ, as expected.

To finish this example, we count the degrees of freedom of the theory before and after the definition of the composite fields in order to make sure that this method consists simply in a change of variables in the functional space of fields. This counting is summarized in table 8.1.

In [MW10], the spinors fields of the SM are also included in the model. The method of change of variables gives rise also to SU(2)-gauge invariant composite fields, as expected by formula (8.1.4).

8.2.2 Application to a $SU(N)$-gauge theory

As an extended model, we apply the previous construction to a $SU(N)$-gauge theory coupled with a scalar field $\phi : \mathcal{M} \to \mathbb{C}^N$. For $N > 2$, it is not possible to clearly identify the dressing field of the theory so that we can't define the corresponding gauge invariant composite field. We explain the reasons of this.

Consider a $SU(N)$-gauge theory for $N > 2$ coupled with a scalar field $\phi : \mathcal{M} \to \mathbb{C}^N$. For $N > 2$, it is not possible to uniquely decompose the field ϕ into the pair (u, η), with

Gauge theory	Electroweak part of the SM (Initial form)	Electroweak part of the SM (Final form)
Gauge group	$\underline{U(1)} \times \underline{SU(2)}$	$\underline{U(1)}$
dimension (1)	$1+3$	1
Fields of the theory	a_μ, b_μ	$A_\mu, W^{\pm}_\mu, Z_\mu$
dimension (2)	$4+3\times 4 = 16$	$4+3\times 4 = 16$
Scalar fields of the theory	Auxiliary field φ	Residual field η
dimension (3)	4	1
Degrees of freedom of the theory (2) + (3) − (1)	16	16

Table 8.1: Fields involved in the theory with their meanings and degrees of freedom, before and after the definition of the composite fields

$u \in SU(N)$ and $\eta = \sqrt{|\phi_1|^2 + \ldots + |\phi_N|^2}$ the norm of ϕ. This can be seen by looking at the degrees of freedom of ϕ before and after such a decomposition. We would obtain

$$\{\text{degrees of } (u,\eta)\} - \{\text{degrees of } \phi\} = (N^2 - 1 + 1) - 2N = N^2 - 2N. \tag{8.2.11}$$

so that this difference gives 0 only for $N = 2$ (and $N = 0$, trivially).

This non-uniqueness of the decomposition of ϕ becomes explicit as we write this scalar field under the form $\phi = \eta u \mathring{\phi}$, where η is the norm of ϕ, u is a $SU(N)$-valued function and $\mathring{\phi}$ is a reference vector in $\mathbb{C}^N$ that we choose equal to $\mathring{\phi} = \begin{pmatrix} 0_{N-1} \\ 1 \end{pmatrix}$, where $\begin{pmatrix} 0_{N-1} \end{pmatrix}$ denotes the null vector for the $N-1$ first components. The norm η is uniquely defined in terms of ϕ but the field $u \in SU(N)$ is only defined *modulo* the left action of elements of the form $\left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where v is a $SU(N-1)$-valued field which acts on the first $N-1$ components of $\mathring{\phi}$. This element v has dimension $(N-1)^2 - 1 = N^2 - 2N$, as in (8.2.11). Then, in this sense, it encodes the obstruction for ϕ to be uniquely decomposed as (u, η). Moreover, for $N = 2$, the element v belongs to $SU(1)$ which, by convention, is the scalar number 1.

One denotes by $[u] : \mathcal{M} \to SU(N)/SU(N-1)$ a field defined on $\mathcal{M}$ with values in the corresponding orbit so that the scalar field ϕ can be decomposed as $(\eta, [u])$. Then, the previous gauge transformation $u^\gamma = \gamma^{-1} u$ is now substituted with the gauge transformation $[u^\gamma] = [\gamma^{-1} u]$ which does not correspond to the correct gauge transformation for the dressing field. In [FFLM13], we show that this method results in a reduction of the gauge group $\underline{SU(N)} \to \underline{SU(N-1)}$.

8.3 Application for YMH theory based on generalized connections

This method is now applied to the YMH type models based on generalized connections on transitive Lie algebroids.

The previous example based on the electroweak part of the Standard Model have allowed to describe the spontaneous symmetry breaking mechanism, usually associated to a dynamical process, in terms of a change of variables in the functional space of fields of the theory. Then, the reduction of the symmetry group of the theory is no more given as a result of a “polarization” process but comes from an appropriate rearrangement of the degrees of freedom of the fields.

This equivalence of description is useful in the context of gauge theories based on generalized connection on A. Indeed, as we have seen in section 7.1, the free parameters of the theory are not running constants related to the energy of the system. They are defined “once for all” in the description of the theory and are related only to the metrics g and h, the dimensions of $\mathcal{M}$ and $\mathcal{L}$ and also on the constant structures C^c_{ab}. Then, a symmetry reduction by changes of variables in the space of fields should be more adapted to this context than a symmetry reduction by means of a spontaneous symmetry breaking.

Since the reduced kernel endomorphism plays a role similar to the scalar field of the BEHHGK mechanism, the dressing field required to define gauge invariant composite fields should be similarly defined out of τ. In the general case, this procedure requires additional investigations on the algebraic structure of τ. In this section, we restrict this procedure to the YMH type model based on a particular subspace of $\Omega^1_{\text{sol}}(\mathsf{A},\mathsf{L})$ defined on an Atiyah Lie algebroid associated to a principal bundle $\mathcal{P}(\mathcal{M},G)$, where G is centerless. The reasons of this specific restriction will be detailed in the following.

The final result of this construction is a YM theory describing massive vector fields A_μ. These vector bosons are composite fields defined for the gauge bosons A_μ and a dressing field u associated to the scalar field τ^b_a. Then, they are invariant with respect to the action of the gauge group $\mathcal{G}$ and ought to correspond to physical observables.

8.3.1 The Yang-Mills-Higgs type model

Consider an orientable Atiyah Lie algebroid over $\mathcal{M}$ associated to the principal bundle $\mathcal{P}(\mathcal{M},G)$, with $G = SU(N)$, equipped with an atlas of Lie algebroids, a Killing inner non-degenerate metric h associated to an inner non-degenerate metric $\widehat{g} = (g, h, \mathring{\nabla})$ on $\Gamma_G(\mathcal{P})$ and a ϕ_{L}-compatible metric on a representation space $\Gamma(\mathcal{E}) = \Gamma(\mathcal{P} \times_\ell \mathbb{C}^N)$.

The Lagrangian we are interested in is the $SU(N)$-gauge theory of section 7.2.2 which describes gauge bosons A_μ and a matter field $s : \mathcal{M} \to \mathbb{C}^N$, interacting with scalar fields $\tau^b_a \in \mathbb{R}$. Here again, to simplify the model, we choose $\mathring{A} = 0$. This theory is then written as

$$\mathcal{L}^{SU(N)}_{Gauge}[A,s,\tau] = -\frac{1}{48}\sum_a f^a_{\mu\nu}\, f^{\mu\nu\, a} + \frac{1}{4n}\sum_{a,b}\mathcal{D}_\mu\tau^b_a\,\mathcal{D}^\mu\tau^b_a - \frac{1}{4\, n(n-1)}\, W(\tau)$$
$$+\frac{1}{4}\sum_a ((\mathcal{D}_\ell)_\mu s)^\dagger\, (\mathcal{D}_\ell)^\mu s - \frac{1}{n}\sum_a (\tau^c_a\, \ell_*(E_c)s)^\dagger\cdot \left(\tau^d_a\, \ell_*(E_d)s\right)$$

where $f^a_{\mu\nu}$ is the field strength associated to the $SU(N)$-connection A_μ, the fields $\tau^b_a : \mathcal{U} \to \mathbb{R}$ are the scalar fields associated to the reduced kernel endomorphism $\tau : \mathsf{L} \to \mathsf{L}$, the covariant derivative of τ^b_a is given as $\mathcal{D}_\mu\tau^a_b = \partial_\mu\tau^b_a + A^{c_1}_\mu\tau^{d_1}_a C^b_{c_1d_1}$, the potential term is written under the form

$$W_\tau = \left(\tau^{d_1}_a\tau^{e_1}_b C^c_{d_1e_1} - C^{d_1}_{ab}\tau^c_{d_1}\right)\left(\tau^{d_2}_a\tau^{e_2}_b C^c_{d_2e_2} - C^{d_2}_{ab}\tau^c_{d_2}\right) \tag{8.3.1}$$

and the covariant derivative of the matter field s is given as $(\mathcal{D}_\ell)_\mu s = \partial_\mu s + \ell_*(A_\mu)s$.

All the fields A, τ and s of the theory support the action of the gauge group $\mathcal{G}$ as

$$\begin{cases} A^g_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g \\ \tau^g = g^{-1}\tau g \\ s^g = \ell(g^{-1})s \end{cases} \tag{8.3.2}$$

where $g : \mathcal{M} \to SU(N)$ is an element of the gauge theory. From the previous section, the theory is invariant with respect to these gauge transformations.

8.3.2 Dressing field for Atiyah Lie algebroid

In the general case, there exists no systematic method to define a dressing field from a tensorial field τ. Then, we restrict the space of generalized connections on Atiyah Lie algebroids to a specific subspace. We denote by $\mathcal{A}_{\mathring{\tau}}$ the space of generalized connections on $\Gamma_G(\mathcal{P})$ such that the reduced kernel endomorphism associated to ϖ can be written as

$$\tau(v) = u\mathring{\tau}(v)u^{-1} \tag{8.3.3}$$

, for any $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$, with $u : \mathcal{P} \to G$ and where $\mathring{\tau} : \mathsf{L} \to \mathsf{L}$ is called the *reference configuration*. We compute the induced right-action of G on the field u and we obtain $u(w{\cdot}g) = g^{-1}u(w)g$, for any $g \in G$ and $w \in \mathcal{P}$. Then, the fields u are elements of the gauge group $\mathcal{G} = \Gamma(\mathcal{P} \times_\alpha G)$, where $\alpha(g)g' = gg'g^{-1}$.

We denote by $\mathcal{G}_{\mathring{\tau}}$ the subgroup of $\mathcal{G}$ associated to the reference configuration $\mathring{\tau}$ which is defined as the set of elements $g \in \mathcal{G}$ such that $g^{-1}\mathring{\tau}(v)g = \mathring{\tau}(v)$, for any $v \in \mathsf{L}$. They are in the "center" of $\mathcal{G}$. The gauge group $\mathcal{G}$ acts on the field τ as $\tau^g = g^{-1}\tau g$, so that it preserves the space $\mathcal{A}_{\mathring{\tau}}$. Then, we define the algebraic action of $\mathcal{G}$ on τ in terms of the field u^g as $\tau^g = u^g\mathring{\tau}u^{g-1}$ where $g \in \mathcal{G}$. Then, we can write

$$g^{-1}u\,\mathring{\tau}\,u^{-1}g = u^g\,\mathring{\tau}\,u^{g-1} \quad \leftrightarrow \quad \mathring{\tau} = (u^{-1}gu^g)\,\mathring{\tau}\,(u^{-1}gu^g)^{-1} \tag{8.3.4}$$

and then, we find that $u^{-1}gu^g$ is an element of $\mathcal{G}_{\mathring{\tau}}$.

To define a non-ambiguous gauge transformation of the field u, we consider the space $\mathcal{A}_{\mathrm{Id}_\mathsf{L}}$ of generalized connections on $\Gamma_G(\mathcal{P})$ whose associated reduced kernel endomorphism can be written as $\tau(v) = \mathrm{Ad}_u v$ for any $v \in \Gamma_G(\mathcal{P}, \mathfrak{g})$ where $u : \mathcal{P} \to G$. We also assume that the group G is centerless so that the subgroup $\mathcal{G}_{\mathrm{Id}_\mathsf{L}}$ is reduced to the space of constant functions with values in e, the identity element of G. Thus, gauge transformations of the field u are uniquely defined as $u^g = g^{-1}u$ and then, the field u is a dressing field as defined in section 8.1.

Let ω be the induced ordinary connection 1-form associated to the generalized connection ϖ on $\Gamma_G(\mathcal{P})$. We define the gauge-invariant composite field $\widehat{\omega}$ associated to the connection ω and the dressing field u as $\widehat{\omega} = u^{-1}\omega u + u^{-1}\widehat{\mathrm{d}}u$, where $\widehat{\mathrm{d}}$ is the graded differential operator acting on $\Omega^\bullet(\mathsf{A}, \mathsf{L})$. Let $s \in \Gamma(\mathcal{E})$, then the composite field $\widehat{s} = \ell(u^{-1}){\cdot}s$ is also gauge invariant. The local expression for the composite field $\widehat{\omega}$ and $\widehat{s}$ are written in terms of the field u_{loc} as

$$\widehat{A}_\mu = u_{\mathrm{loc}}^{-1}A_\mu u_{\mathrm{loc}} + u_{\mathrm{loc}}^{-1}\partial_\mu u_{\mathrm{loc}} \quad ; \quad \widehat{s}_{\mathrm{loc}} = \ell(u_{\mathrm{loc}}^{-1})s_{\mathrm{loc}} \tag{8.3.5}$$

where the local field $u_{\mathrm{loc}} : \mathcal{U} \to G$ is defined by the pull-back of the local cross section $\sigma : \mathcal{U} \to \mathcal{P}_{|\mathcal{U}}$ as $u_{\mathrm{loc}}(p) = u(\sigma(p))$, for any $p \in \mathcal{U}$.

Gauge theory	YMH theory on Atiyah Lie algebroid (Initial form)	YMH theory on Atiyah Lie algebroid (Final form)
Gauge group	$SU(N)$	$\{e\}$
dimension (1)	N^2-1	0
Bosons and fields of the theory	A_μ, s	$\widehat{A}_\mu, \widehat{s}$
dimension (2)	$4\times(N^2-1)+N$	$4\times(N^2-1)+N$
Scalar fields of the theory	τ s.t. $\tau = u^{-1}\mathrm{Id}_{\mathsf{L}} u$	Id_{L}
dimension (3)	N^2-1	0
Degrees of freedom of the theory (2) + (3) − (1)	$4\times(N^2-1)+N$	$4\times(N^2-1)+N$

Table 8.2: Fields involved in the theory with their meanings and degrees of freedom, before and after the definition of the composite fields

8.3.3 YMH model restricted to $\mathcal{A}_{\mathrm{Id}_{\mathsf{L}}}$

We proceed to the change of variables $(A_\mu, s, \tau) \mapsto (\widehat{A}_\mu, \widehat{s}, u)$ so that the gauge theory $\mathcal{L}[A, s, \tau]$ can be written in terms of these new fields as

$$\mathcal{L}^{SU(N)}_{Gauge}[\widehat{A}, \widehat{s}] = \frac{1}{48}\sum_a \widehat{f}^a_{\mu\nu}\,\widehat{f}^{\mu\nu\,a} + \frac{1}{4n}\sum_{a,b} C^b_{ca}C^b_{da}\widehat{A}^c_\mu\widehat{A}^{\mu\,d} + \frac{1}{4}\sum_a \left((\widehat{\mathcal{D}}_\ell)_\mu\widehat{s}\right)^\dagger (\widehat{\mathcal{D}}_\ell)^\mu\widehat{s}$$
$$+\frac{1}{n}\sum_a \left(\ell_*(E_a)s\right)^\dagger\cdot\left(\ell_*(E_a)s\right)$$

where $\widehat{f}^a_{\mu\nu}$ is the field strength associated to the gauge invariant composite field $\widehat{A}_\mu$ and the covariant derivative of the matter field s is given as $(\widehat{\mathcal{D}}_\ell)_\mu = \partial_\mu + \ell_*(\widehat{A}_\mu)$. Using the gauge invariance of the Lagrangian, we see that the theory does not depend explicitly of the u fields but depends exclusively of gauge-invariant fields $\widehat{A}_\mu$ and $\widehat{s}$. Moreover, the space $\mathcal{A}_{\mathrm{Id}_{\mathsf{L}}}$ is a subspace of $\Omega^1_{\mathrm{sol}}(\mathsf{A}, \mathsf{L})$ so that the reduced kernel endomorphism τ associated to this subspace cancels the potential term. Instead of the covariant derivative of τ, we find the new term $\frac{1}{4n}\sum_{a,b} C^b_{ca}C^b_{da}\widehat{A}^c_\mu\widehat{A}^{\mu\,d}$ which correspond in particle physics to a mass term for the vector fields $\widehat{A}_\mu$. The computation of this mass term for low dimensions has been given in section 7.3.

In section 8.2.2, we have seen that the method used to construct gauge invariant composite fields by using a scalar field $\phi : \mathcal{M} \to \mathbb{C}^N$ was possible only for $N = 2$ (actually, this is also possible for $N = 1$). Here, the method can be applied for any N. As in the previous model, the gauge group has not disappeared from the theory. It is still present but its action on the gauge invariant composite field is trivial.The residual field associated to τ is simply Id_{L} so that all the degrees of freedom of τ have been moved to $\widehat{A}_\mu$ and $\widehat{s}$. We summarize our construction in table 8.2

8.4 Commentaries

The method which consists into defining gauge invariant composite fields using some extra degrees of freedom is close to the philosophy of the Goldstone mechanism in particle physics. This mechanism states that, during a spontaneous symmetry breaking, the degrees of freedom, which correspond to the generators of G lost after the reduction, become the so-called *Goldstone bosons*. These bosons are then "eaten" by the gauge bosons A_μ which acquire new degrees of freedom. From massless gauge bosons, with helicity ± 1, they become massive vector fields with spin $\{-1, 0, +1\}$. With regard to this interpretation of the Goldstone mechanism, the mass term of the vector fields is correlated to the reduction of the symmetry group.

In analogy with this mechanism, the method described in this chapter is related to a transfer of degrees of freedom but here, they are carried by the dressing field from a source field, the auxiliary field in our terminology, to a target field, the connection and/or a matter field. Then, composite fields are "bigger", in the sense that they carry additional degrees of freedom, and also gauge-invariant. The auxiliary field, "undressed" of its dressing field, becomes a gauge-invariant residual field. The symmetry group is not reduced in the same sense as in the BEHHGK mechanism. Here, the group still exists but, *de facto*, it is not represented neither on the composite field nor on the residual field. An example given in [FFLM13] is related to the Einstein's equations of gravity described as a geometric gauge theory providing a Cartan connection on a principal bundle. In this example, the residual field is not canonically identified but still, it can be related to a metric on spacetime.

In the YMH model associated to an Atiyah Lie algebroid, there is no dynamical structures which indicate that a spontaneous symmetry breaking can be performed. Indeed, all the free parameters of the theory come either from the geometry or from the algebraic structure of the Atiyah Lie algebroid, and no dynamical parameters are present as in the usual Higgs-like models of the SM. Nevertheless, the extra field τ can actually be used to perform both a symmetry reduction and, simultaneously, an emergence of mass term for the gauge bosons A_μ. It is not possible to do so in the general case, that's why we restrict ourselves to the subspace $\mathcal{A}_{\mathrm{Id}_{\mathsf{L}}}$ of $\Omega^1_{\mathrm{sol}}(\mathsf{A}, \mathsf{L})$. On this subspace, the entire degrees of freedom of τ are contained in the u-field so that, once they are "absorbed" by the connections A_μ and the matter fields s, the Lagrangian does depend neither on u nor on any residual field. It results in the gauge theory with massive vector bosons in interactions with matter fields.

Conclusion

Lie algebroids was initially defined as the infinitesimal version of Lie groupoids [Mac05]. Its formulation in terms of fiber bundles has been widely used in the context of the geometry of Poisson manifolds [KS08; CDW87]. In the present PhD thesis, we rather use the description of Lie algebroids in terms of sections on the vector bundle $\mathcal{A}$ than in terms of geometric objects. This description is close to the formalism of gauge fields theories where gauge fields are defined as sections of associated fiber bundles. Moreover, we know that NCG have established a correspondence between geometric objects and algebraic description. This leads us to consider the description of (transitive) Lie algebroids in terms of $C^\infty(\mathcal{M})$-modules given by the space of sections of the vector bundle $\mathcal{A}$.

As the physics takes place on the base manifold of A, it is necessary to establish local trivializations of transitive Lie algebroids. By doing so, we obtain a supplementary degree of "control" on objects defined on the "top" structure A. Then, the distinction between geometric and algebraic components becomes explicit *e.g.* Lie bracket explicitly defined on $\Gamma(T\mathcal{U}) \oplus \Gamma(\mathcal{U} \times \mathfrak{g})$ by using Lie bracket on $\Gamma(T\mathcal{U})$ and $\mathfrak{g}$, and the representations of vector fields on $\Gamma(\mathcal{U} \times \mathfrak{g})$. Also, local descriptions of A make apparent some Greek and Latin indices related to both the geometry of $\mathcal{M}$ and the Lie algebra $\mathfrak{g}$, respectively. This local description is well-adapted to the formalism of gauge fields theories in physics.

The description of A in terms of sections is compatible with definitions of differential complexes. Indeed, we do not assume the existence of some dual Lie algebroids bundle $\mathcal{A}^*$ to define the space of differential forms on A. In our formulation of transitive Lie algebroids, differential complexes are given in terms of $C^\infty(\mathcal{M})$-linear maps defined on antisymmetric copies of A with values in a given representation space. Associated differential operators are defined using representations of $C^\infty(\mathcal{M})$-modules and the duality of Lie brackets. This "algebraic" description of differential forms on transitive Lie algebroids permits to establish local isomorphisms of differential complexes between $\Omega^\bullet(\mathsf{A})$ (resp. $\Omega^\bullet(\mathsf{A}, \mathsf{L})$) and $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U})$ (resp. $\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})$). Straightforwardly, we show that these isomorphisms are compatible with their associated differential operators.

Within the geometric formalism, transitive Lie algebroids can be considered as a generalization of the tangent bundle $T\mathcal{M}$. In our scheme, these are considered as a generalization of vector fields on $\mathcal{M}$. Indeed, the geometry of vector fields is embedded in A *via* the anchor ρ. Then, the subset of A which can be projected on $\Gamma(T\mathcal{M})$ corresponds to the geometric degrees of freedom of A and the subset of A which does not project on $\Gamma(T\mathcal{M})$, the kernel of ρ, encodes its algebraic degrees of freedom. Contrary to the "geometric part" of A, the "algebraic part" L is defined without ambiguities. In the non-trivial situation, it is not possible to separate these two subspaces. This is analog to vector fields on a principal bundle $\mathcal{P}$: vertical vector fields are uniquely defined on $\mathcal{P}$ but its complementary space in $\Gamma(T\mathcal{P})$ requires an additional connection on $\mathcal{P}$.

The same thing occurs in the context of transitive Lie algebroid. To determine the complementary space of L in A, we use a connection on A which defines an injective $C^\infty(\mathcal{M})$-linear map $\nabla : \Gamma(T\mathcal{M}) \to \mathsf{A}$. Then, this connection ∇ determines the "horizontal" subspace of A in the sense that the map ∇ permits the decomposition $\mathsf{A} = \mathsf{L} \oplus \mathrm{Im}(\nabla)$.

On Atiyah Lie algebroids, this is related to the usual definition of connections defined on principal fiber bundles. This connection is seen as a geometric object in the sense that it does not see the kernel L. However, this geometric description of connection on A can be "bring back" to the algebraic side of A: connections defined on A are equivalently defined by the data of an ordinary connection 1-form $\omega \in \Omega^1(\mathsf{A}, \mathsf{L})$ normalized on L. Here, the normalization of ω on L means $\omega \circ \iota = -\mathrm{Id}_{\mathsf{L}}$. It is a reminiscent of the geometric nature on ∇: it occurs to "kill" the algebraic degrees of freedom of ω so that its dynamical degrees of freedom are only carried by its geometric component. This is obvious as we locally trivialize ω as $\omega_{\text{loc}} = (A^a_\mu \mathrm{d}x^\mu - \theta^a) \otimes E_a$ since all the dynamical degrees of freedom are carried by $A = A^a_\mu \mathrm{d}x^\mu \otimes E_a$ and θ^a is an element of $\mathfrak{g}^*$. On Atiyah Lie algebroids, ordinary connection 1-form are in 1:1 correspondence with Ehresmann connections 1-form on $\mathcal{P}$ [LM12a]. Covariant derivatives and curvatures inherits the "all-geometric" nature of the connection 1-form: with respect to the Cartan operation $(\mathsf{L}, i, \mathsf{L})$, these differential forms are L-horizontal and their local trivializations depend only on the geometric connection $A \in \Omega^1(\mathcal{U}, \mathsf{L})$.

Actually, this "all-geometric" nature of ω is encoded in its normalization constraint on L. Then, generalized connections defined on A can be considered as perturbations of ω in $\Omega^1(\mathsf{A}, \mathsf{L})$ where the object $\tau : \mathsf{L} \to \mathsf{L}$ parametrizes this perturbation so that we have $\omega \circ \iota = -\mathrm{Id}_{\mathsf{L}} + \tau$. Conversely, we define the space of generalized connection on A as the space $\Omega^1(\mathsf{A}, \mathsf{L})$ and we denote by τ the measure of the obstruction for a generalized connection ϖ to be normalized on L. For these two definitions, the parameter τ encodes the "non-geometric nature" of ϖ. We make it apparent as we decompose any generalized connection ϖ into the sum $\omega + \tau(\mathring{\omega})$, where ω is the induced ordinary connection 1-form associated to ϖ and $\mathring{\omega}$ is a background connection on A. This sum represents, *modulo* the background connection, the separation of the geometric degrees of freedom and the algebraic degrees of freedom of ϖ. Once locally trivialized, these two components can be considered as independents gauge fields. This independence is also apparent from the point of view of the gluing transformations by changes of trivializations.

The expression of generalized covariant derivatives in terms of ω and τ makes apparent an algebraic extension of the "usual" geometric covariant derivative. This extension contributes in some minimal coupling between the "parameter" τ and some spaces of representation. The curvature of generalized connection is defined with the Cartan structure equation as $\widehat{\mathrm{d}}\varpi + \frac{1}{2}[\varpi, \varpi]$. In terms of ω and τ, the expression of this curvature is more cumbersome. An adapted formulation makes apparent the decomposition of $\widehat{F}$ in the so-called mixed basis as the sum of three terms. The first term corresponds to the purely geometric component of $\widehat{F}$, written in terms of the curvatures of ω and $\mathring{\omega}$. The second term mixes geometric and algebraic degrees of forms under the form of a covariant derivative term. Finally, the last term is purely agebraic and is interpreted as the obstruction for τ to preserve the Lie bracket on L.

The bi-nature (geometric and algebraic) of generalized connections leads to some incompatibility relations with respect to gauge transformations. Even if the "usual" infinitesimal gauge transformations, given by the Lie derivative along L, are well-defined on generalized connections, they result in messy gauge transformations. For example, covariant derivatives associated to ϖ, acting on a representation space $\mathcal{E}$, do not preserve the representation of L on $\Gamma(\mathcal{E})$ *i.e.* we obtain $\widehat{\mathcal{D}}^\xi s^\xi \neq (\widehat{\mathcal{D}} s)^\xi$. Moreover, gauge transformations of the curvature associated to ϖ are awful to manipulate in order to construct gauge invariant quantities. These results come from the fact that "usual" gauge transformations are actually *geometric* transformations. Indeed, on Atiyah Lie algebroids, the gauge action of L is given by the Lie derivative, is exactly the infinitesimal version of the action of the gauge group $\mathcal{G}$ by vertical automorphisms. Then, these geometric gauge transformations cannot be compatible with the new algebraic degrees of freedom of ϖ. A new gauge action of L have been defined specially for generalized connections in order to obtain convenient gauge transformations more manageable to construct gauge-invariant quantities. These new gauge transformations and the old "usual" ones coincide on the space on ordinary connection 1-forms. This scheme is also present in the works of T. Masson in NCG (see [Mas] and references therein) .

Gauge invariant action functionals are defined as the "norm" of the curvature associated to a generalized connection ϖ. Lagrangians $\mathcal{L}[\varpi]$ associated to these action functionals are defined by the composition of two operations. Starting from $\widehat{F} \in \Omega^2(\mathsf{A}, \mathsf{L})$, we use an inner non-degenerate metric $\widehat{g} = (g, h, \mathring{\nabla})$, where h is a Killing inner, and a Hodge star operator on A to build a differential form $h(\widehat{R}, \star \widehat{R})$ with degree $(m+n)$ defined on A with values in $C^\infty(\mathcal{M})$. Then, the inner integral "extracts" the component of this differential form which is factorized by the volume form ω_{vol}. It results in a m-form defined on vector fields on $\mathcal{M}$. As a m-form, this object can be written as $\mathcal{L}[\varpi]\text{dvol}$, where dvol denotes the volume form on $\mathcal{M}$. It results in the expression of $\mathcal{L}[\varpi] \in C^\infty(\mathcal{M})$ as the sum of the square of each components associated to the decomposition of $\widehat{F}$ in terms of ω and τ.

The Lagrangian associated to a generalized connection form a YMH type model, depicted in a background reference, where the local expression of τ plays the role of the scalar field of the BEHHGK mechanism of spontaneous symmetry breaking [PS95; IZ85]. The potential term for τ is not related to any dynamical parameters which would modify its "shape", depending on the energy of the system. Instead, it is defined as an algebraic constraint on τ. It is obvious that the usual spontaneous symmetry breaking mechanism of the SM makes few sense in this context. There, we have used an alternative method of symmetry reduction which has consisted into a change of variables in the functional space of fields (A, τ). In analogy the Goldstone mechanism, degrees of freedom of τ which are in the representation of the gauge group are moved to the gauge connections A in order to create gauge-invariant composite fields. Presently, this method cannot be performed in the general case. That's why we have applied this method only for generalized connections on Atiyah Lie algebroids with fields τ which can be written as Ad_u, where $u \in \mathcal{G}$. The counting of the theory's degrees of freedom, before and after this operation, leads to consider this method merely as a change of variables. It results in a YM theory describing massive vector bosons A_μ, with the action of the symmetry group $SU(N)$ reduced to the neutral element $\{e\}$.

All these results form an essential contribution to constructions of gauge field theories and, in particular, to the definition of YMH type models which are directly relevant to the SM. A similar result have been obtained from the NCG of A. Connes [Con94; CM08a; CCM07] but here, the formalism of transitive Lie algebroids admits representation of Lie groups, not only modules. Moreover, the description of transitive Lie algebroids in terms of sections, rather than in terms of fiber bundles, gives more manageable constructions. These works should be soon extended by additional constructions coming either from the formalism of gauge fields theory or from transitive Lie algebroids themselves. The next three points establishes my research projects for my future post-doctoral positions.

In the context of gauge field theories, we have to explore the renormalization of $SU(N)$-gauge theories of YMH type based on transitive Lie algebroids. Here, we forget about the geometric and algebraic scheme of the theory of Lie algebroids and we simply consider the Lagrangian $\mathcal{L}^{SU(N)}_{gauge}[A, \tau, \mathring{A}, s]$ of section 7.2.2 as a gauge theory toy-model related to the

SM. We need only to employ the usual tools of the QFT so that, starting with the classical Lagrangian, we compute the vertexes associates to the Feynman's diagrams of the gauge fields A_μ and τ_a^b in order to study the renormalization of the theory at the 1-loop order.

The method of symmetry reduction by change of variables described in chapter 8 have been applied to a "convenient" subset of generalized connections on Atiyah Lie algebroids. This method have shown that the field τ plays an important role in the construction by "moving" its gauge degrees of freedom of τ to the induced ordinary connection in order to get a gauge invariant theory with massive vector fields. In the more general case, does it exist a relation between the reduction of the group of symmetry and the algebraic properties of τ? I expect that these works will allow to confirm the analogy between τ and the scalar field ϕ of the BEHHGK mechanism.

We have incorporated scalar fields in YMH type theories associated to generalized connections. What about spinor fields which describe fermionic matter? From the point of view of mathematical physics, Dirac spinors are in a representation of a Clifford algebra associated to the vector space $\Gamma(T\mathcal{M})$ and a metric g on $\mathcal{M}$. Besides, transitive Lie algebroids are considered as generalizations of vector fields on $\mathcal{M}$ and are equipped with a metric $\widehat{g}$. Moreover, if the metric $\widehat{g}$ is an inner non degenerate metric, then there exists a unique connection $\mathring{\nabla}$ on A so that the transitive Lie algebroids can be written as $\mathsf{A} = \mathsf{L} \oplus \mathrm{Im}(\mathring{\nabla})$ and the metric $\widehat{g}$ is block-diagonal with respect to this decomposition. Then, this result should be employed to extend the usual Dirac operator of the field theory by an algebraic element coming from the fiber $\mathcal{L}$ of the Lie algebroid. Applied to gauge field theory based on transitive Lie algebroids, I expect to extend the usual spinor fields of the SM by some algebraic objects which should bring new phenomenological interpretations in particle physics.

Appendices

Appendix A

Čech-de Rham bicomplex

A.1 Recalls

A.1.1 Partition function of the unity

The *partition function of the unity* is defined as follows. Given an atlas of $\mathcal{M}$, a partition function of the unity is a collection of $C^\infty(\mathcal{M})$ maps $(\rho_i)_{i\in I}$, such that, for any point $p \in \mathcal{M}$, one has:

- $\rho_i(p) > 0$ for any $p \in \mathcal{U}_i$ for any $i \in I$.
- ρ_i vanishes outside of $\mathcal{U}_i$.
- $\sum_{i\in I} \rho_i(p) = 1$ for any $p \in \mathcal{M}$.

A.1.2 Čech cohomology

Let $\mathcal{M}$ be a manifold covered by opens $(\mathcal{U}_i)_{i\in I}$, sorted in the growing order $(\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_I)$. Let $\mathfrak{F}$ be a *functor* which assign to any open $\mathcal{U}_i$ a differential form defined on $\mathcal{U}_i$. Let $C^p(\mathcal{U}, \mathfrak{F})$ defined as:

$$C^p(\mathcal{U}, \mathfrak{F}) = \prod_{\alpha_1<\ldots<\alpha_q} \mathfrak{F}(\mathcal{U}_{\alpha_1\alpha_2\ldots\alpha_p}) \tag{A.1.1}$$

where $\mathcal{U}_{\alpha_1\alpha_2\ldots\alpha_p} = \mathcal{U}_{\alpha_1} \cap \mathcal{U}_{\alpha_2} \cap \ldots \cap \mathcal{U}_{\alpha_p}$. The element $C^0(\mathcal{U}, \mathfrak{F}) = C^\infty(\mathcal{M})$. An element $\omega \in C^p(\mathcal{U}, \mathfrak{F})$ is a collection of forms defined on $\mathcal{U}_{\alpha 1\ldots\alpha_p}$. The map $\omega_{\alpha_1\alpha_2\ldots\alpha_{p-1}}$ denotes the representative of ω on the intersection $\mathcal{U}_{\alpha_1\alpha_2\ldots\alpha_{p-1}}$. The differential operator $\delta : C^p(\mathcal{U}, \mathfrak{F}) \to C^{p+1}(\mathcal{U}, \mathfrak{F})$ is defined as:

$$(\delta\omega)_{\alpha_1\alpha_2\ldots\alpha_{p+1}} = \sum_{i=1}^{p+1} (-1)^{i+1} \omega_{\alpha_1\alpha_2\ldots\hat{\alpha}_i\ldots\alpha_{p+1}|\alpha_1\ldots\alpha_{p+1}} \tag{A.1.2}$$

where $\omega_{\alpha_1\alpha_2\ldots\hat{\alpha}_i\ldots\alpha_{p+1}|\alpha_1\ldots\alpha_{p+1}}$ is restricted to the open $\mathcal{U}_{\alpha_1\alpha_2\ldots\alpha_{p+1}}$. It is straightforward to show that $\delta \circ \delta = 0$. One has the following long sequence of differential complex:

$$0 \longrightarrow C^0(\mathcal{U}, \mathfrak{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathfrak{F}) \xrightarrow{\delta} \ldots \xrightarrow{\delta} C^q(\mathcal{U}, \mathfrak{F}) \xrightarrow{\delta} \ldots \tag{A.1.3}$$

The *Čech cohomology* is the cohomological sequence $\mathrm{H}^\bullet_{\mathrm{dR}}(\mathcal{U}, \mathfrak{F}) = \bigoplus_{q=0} \mathrm{H}^q_{\mathrm{dR}}((\mathcal{U}, \mathfrak{F}), \delta)$ where

$$\mathrm{H}^q(\mathcal{U}, \mathfrak{F}, \delta) = \mathrm{Z}^q((\mathcal{U}, \mathfrak{F}), \delta)/\mathrm{B}^q((\mathcal{U}, \mathfrak{F}), \delta). \tag{A.1.4}$$

where $\mathrm{Z}^q((\mathcal{U}, \mathfrak{F}), \delta)$ denotes the space of closed q-forms and $\mathrm{B}^q((\mathcal{U}, \mathfrak{F}), \delta)$ denotes the space of exact q-forms

A.2 Generalized Čech-de Rham bicomplex

Let $(\mathcal{U}_1, S_1)$ and $(\mathcal{U}_2, S_2)$ such that $\mathcal{U}_2 \subset \mathcal{U}_1$ be two local trivializations of $\mathsf{A}_{|\mathcal{U}_1}$. For any $\omega_1 \in \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_1, \mathfrak{g})$, one denotes by $\omega_{1|2}$ its restriction to $\mathcal{U}_2$.

Denote by $\widehat{\alpha}_1^2$ the map defined as in (3.3.8) over the open $\mathcal{U}_1 \cap \mathcal{U}_2 = \mathcal{U}_2$ (here, $\mathcal{U}_{12} = \mathcal{U}_2$). One defines the map

$$i_{\mathcal{U}_1}^{\mathcal{U}_2} : \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_1, \mathfrak{g})_{|\mathcal{U}_2} \to \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_2, \mathfrak{g}), \quad ; \quad i_{\mathcal{U}_1}^{\mathcal{U}_2}(\omega_{1|2}) = \widehat{\alpha}_1^2(\omega_{1|2}). \tag{A.2.1}$$

One establishes the following properties of the map i

$$i_{\mathcal{U}_1}^{\mathcal{U}_1}(\omega_1) = \omega_1 \quad ; \quad i_{\mathcal{U}_2}^{\mathcal{U}_1} \circ i_{\mathcal{U}_1}^{\mathcal{U}_2}(\omega_{1|2}) = (\omega_{1|2}) \quad ; \quad i_{\mathcal{U}_2}^{\mathcal{U}_3} \circ i_{\mathcal{U}_1}^{\mathcal{U}_2}(\omega_{1|3}) = i_{\mathcal{U}_1}^{\mathcal{U}_3}(\omega_{1|3}) \tag{A.2.2}$$

for any $\mathcal{U}_3 \subset \mathcal{U}_2 \subset \mathcal{U}_1$.

One denotes by $\mathfrak{F}$ the functor which assign to any open $\mathcal{U}_i$ with $i \in I$ a map $\mathcal{U}_i \mapsto \Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$ which is a presheaf of graded differential Lie algebras.

Let A be a transitive Lie algebroid over $\mathcal{M}$ with kernel L equipped with a atlas of Lie algebroids $(\mathcal{U}_i, \Psi_i, \nabla_i^0)_{i\in I}$. Let $(\mathcal{U}_i)_{i\in I}$ be a good cover of $\mathcal{M}$ whose opens are sorted in the growing order $(\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_I)$. On any $\mathcal{U}_{\alpha_0\dots\alpha_p} = \mathcal{U}_{\alpha_0} \cap \dots \cap \mathcal{U}_{\alpha_p} \neq \varnothing$, one assumes there is a local trivialization of $\mathsf{A}_{|\mathcal{U}_{\alpha_0\dots\alpha_p}}$.

Assume that to any $\mathcal{U}_{\alpha_0\dots\alpha_p} \neq \varnothing$, one associates a differential complex which one denotes by $\Omega^\bullet_{\mathsf{TLA}}(U_{\alpha_0\dots\alpha_p}, \mathfrak{g})$. These differential complexes *are not necessarily* the corresponding trivializations of $\Omega^q(\mathsf{A}, \mathsf{L})$ over $\mathcal{U}_{\alpha_0\dots\alpha_p}$. One defines the map

$$\widehat{\alpha}_{\alpha_0\dots\alpha_p}^{\alpha_k\alpha_0\dots\alpha_p} : \Omega^\bullet(\mathcal{U}_{\alpha_0\dots\alpha_p}, \mathfrak{g})_{|\mathcal{U}_{\alpha_k\alpha_0\dots\alpha_p}} \to \Omega^\bullet(\mathcal{U}_{\alpha_k\alpha_0\dots\alpha_p}, \mathfrak{g}). \tag{A.2.3}$$

We define an extended version of the ordinary Čech-de Rham bicomplex associated to the presheaf $\mathfrak{F}$. Consider the bicomplex

$$C^{p,q}(\mathcal{U}, \mathfrak{F}) = C^p(\mathcal{U}, \Omega^q_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g})) = \prod_{\alpha_0 < \dots < \alpha_p} \Omega^q_{\mathsf{TLA}}(\mathcal{U}_{\alpha_0\dots\alpha_p}, \mathfrak{g}) \tag{A.2.4}$$

One denotes by p and q the *Čech degree* and the *de Rham degree* of $C^{p,q}(\mathcal{U}, \mathfrak{F})$, respectively. We denote by $\omega_{\alpha_0\cdots\alpha_p} \in C^p(\mathcal{U}, \Omega^q_{\mathsf{TLA}}(\mathcal{U}, \mathfrak{g}))$ an element (p, q) of this bicomplex. For $p = -1$, one defines $C^{-1,q}(\mathcal{U}, \mathfrak{F}) = \Omega^q(\mathsf{A}, \mathsf{L})$. For any (p, q), one defines the map $\widetilde{\delta}$ as follows:

$$\widetilde{\delta} : C^{p,q}(\mathcal{U}, \mathfrak{F}) \to C^{p+1,q}(\mathcal{U}, \mathfrak{F}) \quad ; \quad (\widetilde{\delta}\omega)_{\alpha_0\dots\alpha_{p+1}} = \sum_{k=0}^{p} (-1)^k i_{\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}}^{\mathcal{U}_{\alpha_0\dots\alpha_{p+1}}}(\omega_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}|\alpha_0\dots\alpha_{p+1}}) \tag{A.2.5}$$

with $\omega_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}|\alpha_0\dots\alpha_{p+1}} \in \Omega^q(\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}, \mathfrak{g})_{|\mathcal{U}_{\alpha_0\dots\alpha_{p+1}}}$ where $\alpha_0 \dots \cancel{\alpha_k} \dots \alpha_{p+1}$ means the omission of the index α_k. Here the map $i_{\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}}^{\mathcal{U}_{\alpha_0\dots\alpha_{p+1}}}$ is defined as

$$i_{\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}}^{\mathcal{U}_{\alpha_0\dots\alpha_{p+1}}}(\omega_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}|\alpha_0\dots\alpha_{p+1}}) = \widehat{\alpha}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}^{\alpha_0\dots\alpha_{p+1}}(\omega_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}|\alpha_0\dots\alpha_{p+1}}) \tag{A.2.6}$$

The map $\widetilde{\delta} : C^{-1,q}(\mathcal{U}, \mathfrak{F}) \to C^{0,q}(\mathcal{U}, \mathfrak{F})$ is defined as the local trivialization of $\Omega^q(\mathsf{A}, \mathsf{L})$ with respect to the atlas $(\mathcal{U}_i, S_i)_{i\in I}$. One adopts the Čech convention:

$$i_{\mathcal{U}_{\alpha_0\dots\alpha_i\dots\alpha_j\dots\alpha_p}}^{\mathcal{U}_{\alpha_0\dots\alpha_p}} = -i_{\mathcal{U}_{\alpha_0\dots\alpha_j\dots\alpha_i\dots\alpha_p}}^{\mathcal{U}_{\alpha_0\dots\alpha_p}} \tag{A.2.7}$$

for any i, j. This implies that the element $\omega_{\alpha_0 \ldots \alpha_p}$ is a completely antisymmetric tensor object.

For any $\omega \in \Omega^{\bullet}_{\mathsf{TLA}}(\mathcal{U}_{\alpha_0 \ldots \alpha_{p-1}}, \mathfrak{g})$, one uses the chain relation on $i^{\mathcal{U}_j}_{\mathcal{U}_i}$ in order to compute:

$$
\begin{aligned}
(\widetilde{\delta}(\widetilde{\delta}\omega))_{\alpha_0 \ldots \alpha_{p+1}} &= \sum_{k=0}^{p} (-1)^k i^{\mathcal{U}_{\alpha_0 \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}} ((\widetilde{\delta}\omega)_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1} | \alpha_0 \ldots \alpha_{p+1}}) \\
= \sum_{k=0}^{p} (-1)^k i^{\mathcal{U}_{\alpha_0 \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}} \Big(& \sum_{\substack{k'=0 \\ k'<k}}^{p} (-1)^{k'} i^{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_{k'}} \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}} (\omega_{\alpha_0 \ldots \cancel{\alpha_{k'}} \ldots \cancel{\alpha_k} \ldots \alpha_{p+1} | \alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}) \\
&+ \sum_{\substack{k'=0 \\ k'>k}}^{p} (-1)^{k'+1} i^{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \cancel{\alpha_{k'}} \ldots \alpha_{p+1}}} (\omega_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \cancel{\alpha_{k'}} \ldots \alpha_{p+1} | \alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}) \Big) \\
&= \sum_{k=0}^{p} \sum_{\substack{k'=0 \\ k'<k}}^{p} (-1)^{k+k'} i^{\mathcal{U}_{\alpha_0 \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_{k'}} \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}} (\omega_{\alpha_0 \ldots \cancel{\alpha_{k'}} \ldots \cancel{\alpha_k} \ldots \alpha_{p+1} | \alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}) \\
&+ \sum_{k=0}^{p} \sum_{\substack{k'=0 \\ k'>k}}^{p} (-1)^{k+k'+1} i^{\mathcal{U}_{\alpha_0 \ldots \alpha_{p+1}}}_{\mathcal{U}_{\alpha_0 \ldots \cancel{\alpha_{k'}} \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}}} (\omega_{\alpha_0 \ldots \cancel{\alpha_k} \ldots \cancel{\alpha_{k'}} \ldots \alpha_{p+1} | \alpha_0 \ldots \cancel{\alpha_k} \ldots \alpha_{p+1}})
\end{aligned}
\tag{A.2.8}
$$

It is straightforward to show that the sum of these two last terms give zero. Then one has:

$$\widetilde{\delta} \circ \widetilde{\delta} = 0 \tag{A.2.9}$$

Let $\mathcal{U}_i$ and $\mathcal{U}_j$ such that $\mathcal{U}_j \subset \mathcal{U}_i$. For any $\omega \in \Omega^q_{\mathsf{TLA}}(\mathcal{U}_i, \mathfrak{g})$ and using the formula (2.2.9), one computes:

$$
\begin{aligned}
&(\widehat{\mathrm{d}}_{\mathsf{TLA}}(\widetilde{\delta}\omega)_j)(X_1 \oplus \gamma_1, \ldots, X_{q+1} \oplus \gamma_{q+1}) \\
&\quad = \sum_{k}^{q+1} (-1)^{k+1} X_i \cdot (\widetilde{\delta}\omega)_j (X_1 \oplus \gamma_1, \ldots, \overset{k}{\vee}, \ldots, X_{q+1} \oplus \gamma_{q+1}) \\
&\quad + \sum_{k}^{q+1} (-1)^{k+1} [\gamma_k, (\widetilde{\delta}\omega)_j (X_1 \oplus \gamma_1, \ldots, \overset{k}{\vee}, \ldots, X_{q+1} \oplus \gamma_{q+1})] \\
&+ \sum_{1 \leq k < k' \leq q+1} (-1)^{k+k'} (\widetilde{\delta}\omega)_j ([X_k \oplus \gamma_k, X_{k'} \oplus \gamma_{k'}], X_1 \oplus \gamma_1, \ldots, \overset{k}{\vee}, \ldots, \overset{k'}{\vee}, \ldots, X_{q+1} \oplus \gamma_{q+1}) \\
&\quad = \sum_{k}^{q+1} (-1)^{k+1} X_i \cdot (\alpha_{ji} \circ \omega_{i|j}) (S^i_j(X_1 \oplus \gamma_1), \ldots, \overset{k}{\vee}, \ldots, S^i_j(X_{q+1} \oplus \gamma_{q+1})) \\
&\quad + \sum_{k}^{q+1} (-1)^{k+1} [\gamma_k, (\alpha_{ji} \circ \omega_{i|j}) (S^i_j(X_1 \oplus \gamma_1), \ldots, \overset{k}{\vee}, \ldots, S^i_j(X_{q+1} \oplus \gamma_{q+1}))] \\
&\quad + \sum_{1 \leq k < k' \leq q+1} (-1)^{k+k'} (\alpha_{ji} \circ \omega_{i|j}) (S^i_j([(X_k \oplus \gamma_k), (X_{k'} \oplus \gamma_{k'})]), \ldots, \overset{k}{\vee}, \ldots, \overset{k'}{\vee}, \ldots)
\end{aligned}
$$

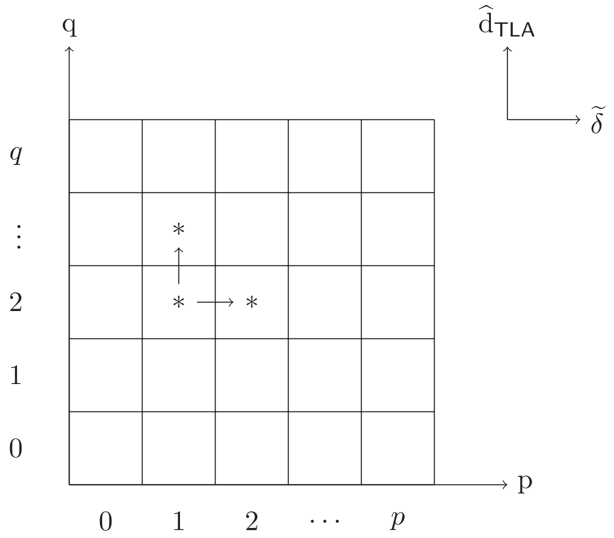
$$
\begin{aligned}
&= \sum_{k}^{q+1}(-1)^{k+1}\alpha_{ji}\circ(X_i\cdot\omega_{i|j})(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))\\
&- \sum_{k}^{q+1}(-1)^{k+1}[\chi_{ji}(X),(\alpha_{ji}\circ\omega_{i|j})(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))]\\
&+ \sum_{k}^{q+1}(-1)^{k+1}[\alpha_{ji}(\gamma_k),(\alpha_{ji}\circ\omega_{i|j})(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))]\\
&+ \sum_{k}^{q+1}(-1)^{k+1}[\chi_{ji}(X),(\alpha_{ji}\circ\omega_{i|j})(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))]\\
&+ \sum_{1\le k<k'\le q+1}(-1)^{k+k'}(\alpha_{ji}\circ\omega_{i|j})(S^i_j([(X_k\oplus\gamma_k),(X_{k'}\oplus\gamma_{k'})]),\ldots,\overset{k}{\vee},\ldots,\overset{k'}{\vee},\ldots)\\
&= \sum_{k}^{q+1}(-1)^{k+1}\alpha_{ji}\circ(X_i\cdot\omega_{i|j})(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))\\
&+ \sum_{k}^{q+1}(-1)^{k+1}\alpha_{ji}\circ([\gamma_k,\omega_{i|j}(S^i_j(X_1\oplus\gamma_1),\ldots,\overset{k}{\vee},\ldots,S^i_j(X_{q+1}\oplus\gamma_{q+1}))]\\
&+ \sum_{1\le k<k'\le q+1}(-1)^{k+k'}(\alpha_{ji}\circ\omega_{i|j})([S^i_j(X_k\oplus\gamma_k),S^i_j(X_{k'}\oplus\gamma_{k'})]),\ldots,\overset{k}{\vee},\ldots,\overset{k'}{\vee},\ldots)\\
&= \widehat{\alpha}^j_i(\widehat{\mathrm{d}}_{\mathsf{TLA}}\omega)_{i|j}(X_1\oplus\gamma_1,\ldots,X_{q+1}\oplus\gamma_{q+1})\\
&= \widetilde{\delta}(\widehat{\mathrm{d}}_{\mathsf{TLA}}\omega)_j(X_1\oplus\gamma_1,\ldots,X_{q+1}\oplus\gamma_{q+1})
\end{aligned}
$$

Then one has the following result:

$$
\widehat{\mathrm{d}}_{\mathsf{TLA}}\circ\widetilde{\delta}=\widetilde{\delta}\circ\widehat{\mathrm{d}}_{\mathsf{TLA}} \tag{A.2.10}
$$

for any opens $\mathcal{U}_i$ and $\mathcal{U}_j$ such that $\mathcal{U}_j\subset\mathcal{U}_i$. The map $i^{\mathcal{V}}_{\mathcal{U}}$ with $\mathcal{V}\subset\mathcal{U}$ is a morphism of graded differential Lie algebras.

The Čech-de Rham bicomplex associated to the presheaf $\mathfrak{F}$ is summarize in the following table.



One has the long sequence

$$0 \longrightarrow C^{-1,q}(\mathcal{U},\mathfrak{F}) \xrightarrow{\widetilde{\delta}} C^{0,q}(\mathcal{U},\mathfrak{F}) \xrightarrow{\widetilde{\delta}} C^{1,q}(\mathcal{U},\mathfrak{F}) \xrightarrow{\widetilde{\delta}} C^{2,q}(\mathcal{U},\mathfrak{F}) \xrightarrow{\widetilde{\delta}} \dots \tag{A.2.11}$$

In order to prove the exactness of this long sequence, we introduce a partition function of unity as defined in A.1.1 $\{\rho_i\}_{i\in I}$ subordinated to the good cover $\{U_i\}_{i\in I}$. Let $\omega \in C^{p,q}(\mathcal{U},\mathfrak{F})$ an exact form *i.e.* such that

$$(\widetilde{\delta}\omega)_{\alpha_0\dots\alpha_{p+1}} = \sum_{k=0}^{p}(-1)^k i^{\mathcal{U}_{\alpha_0\dots\alpha_{p+1}}}_{\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_{p+1}}} \Big((\omega_{\alpha_0\cdots\cancel{\alpha_k}\dots\alpha_{p+1}})|_{\alpha_0\dots\alpha_{p+1}}\Big) = 0. \tag{A.2.12}$$

For any $\omega_{\alpha_0\dots\alpha_p} \in \Omega^q_{\mathsf{TLA}}(\mathcal{U}_{\alpha_0\dots\alpha_p},\mathfrak{g})$, the element $\rho_{\alpha_k}\cdot\omega_{\alpha_0\dots\alpha_p}$ is assumed to be defined on $\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}$ where ρ_{α_k} comes out from the partition function of the unity of $\mathcal{M}$ for any $1 < k < p$. For any $\alpha_0 < \dots < \alpha_p$, let us define

$$\tau_{\alpha_0\dots\alpha_p} = \sum_{\alpha_k=1}^{I} \widehat{\alpha}^{\alpha_0\dots\alpha_p}_{\alpha_k\alpha_0\dots\alpha_p}\left(\rho_{\alpha_k}\cdot\omega_{\alpha_k\alpha_0\dots\alpha_p}\right)$$

A straightforward computation shows that

$$\begin{aligned}(\widetilde{\delta}\tau)_{\alpha_0\dots\alpha_p} &= \sum_{k=0}^{p}(-1)^k i^{\mathcal{U}_{\alpha_0\dots\alpha_p}}_{\mathcal{U}_{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}}\Big(\tau_{\alpha_0\cdots\cancel{\alpha_k}\dots\alpha_p|\alpha_0\dots\alpha_p}\Big)\\ &= \sum_{k=0}^{p}(-1)^k\widehat{\alpha}^{\,\alpha_0\dots\alpha_p}_{\,\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}\left(\left(\sum_{\alpha_{k'}=1}^{I}\widehat{\alpha}^{\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}_{\alpha_{k'}\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}\Big(\rho_{\alpha_{k'}}\cdot\omega_{\alpha_{k'}\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}\Big)\right)|_{\alpha_0\dots\alpha_p}\right)\end{aligned}$$

Since ω is an exact form, one has the relation:

$$\widehat{\alpha}^{\,\alpha_{k'}\alpha_0\dots\alpha_p}_{\,\alpha_0\dots\alpha_p}\Big(\omega_{\alpha_0\dots\alpha_p|\alpha_{k'}\alpha_0\dots\alpha_p}\Big) + \sum_{k=0}^{p}(-1)^{k+1}\widehat{\alpha}^{\,\alpha_{k'}\alpha_0\dots\alpha_p}_{\,\alpha_{k'}\alpha_0\dots\cancel{\alpha_k}\dots\alpha_p}\Big(\omega_{\alpha_{k'}\alpha_0\cdots\cancel{\alpha_k}\dots\alpha_p|\alpha_{k'}\alpha_0\dots\alpha_p}\Big) = 0. \tag{A.2.13}$$

So that,

$$
\begin{aligned}
(\widetilde{\delta}\tau)_{\alpha_0\ldots\alpha_p} &= \sum_{k=0}^{p}(-1)^k \widehat{\alpha}\,^{\alpha_0\ldots\alpha_p}_{\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}\left(\left(\sum_{\alpha_{k'}=1}^{I}\widehat{\alpha}\,^{\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}\left(\rho_{\alpha_{k'}}\cdot\omega_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}\right)\right)|_{\alpha_0\ldots\alpha_p}\right)\\
&= \sum_{\alpha_{k'}=1}^{I}\sum_{k=0}^{p}(-1)^k \widehat{\alpha}\,^{\alpha_0\ldots\alpha_p}_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}\left((\rho_{\alpha_{k'}}\cdot\omega_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p})|_{\alpha_0\ldots\alpha_p}\right)\\
&= \sum_{\alpha_{k'}=1}^{I}\sum_{k=0}^{p}(-1)^k \rho_{\alpha_{k'}}\cdot\widehat{\alpha}\,^{\alpha_0\ldots\alpha_p}_{\alpha_{k'}\alpha_0\ldots\alpha_p}\widehat{\alpha}\,^{\alpha_{k'}\alpha_0\ldots\alpha_p}_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}\left(\omega_{\alpha_{k'}\alpha_0\ldots\cancel{\alpha_k}\ldots\alpha_p}|_{\alpha_0\ldots\alpha_p}\right)\\
&= \sum_{\alpha_{k'}=0}^{I}\rho_{\alpha_{k'}}\widehat{\alpha}\,^{\alpha_0\ldots\alpha_p}_{\alpha_{k'}\alpha_0\ldots\alpha_p}\widehat{\alpha}\,^{\alpha_{k'}\alpha_0\ldots\alpha_p}_{\alpha_0\ldots\alpha_p}(\omega_{\alpha_0\ldots\alpha_p})\\
&= \sum_{\alpha_{k'}=0}^{I}\rho_{\alpha_{k'}}(\omega_{\alpha_0\ldots\alpha_p})\\
&= \omega_{\alpha_0\ldots\alpha_p}
\end{aligned}
$$

This proves that any exact form ω is closed. Then, the long exact sequence (A.2.11) is exact.

One denotes by $K^\bullet(\mathcal{U},\mathfrak{F}) = \bigoplus_{r=0} K^r(\mathcal{U},\mathfrak{F})$ the total bicomplex where :

$$K^r(\mathcal{U},\mathfrak{F}) = \bigoplus_{p+q=r} C^{p,q}(\mathcal{U},\mathfrak{F}) \tag{A.2.14}$$

This total bicomplex is equipped with the differential operator:

$$D : K^r(\mathcal{U},\mathfrak{F}) \to K^{r+1}(\mathcal{U},\mathfrak{F}) \quad ; \quad D = \widehat{\mathrm{d}}_{\mathsf{TLA}} + (-1)^p\widetilde{\delta} \tag{A.2.15}$$

where p denotes the Čech degree of $K^r(\mathcal{U},\mathfrak{F})$. It is easy to check that following properties:

$$D\circ D = 0 \quad ; \quad \widetilde{\delta}\circ\widehat{\mathrm{d}} = D\circ\widetilde{\delta} \quad ; \quad \widetilde{\delta}\circ\widehat{\mathrm{d}} = \widehat{\mathrm{d}}_{\mathsf{TLA}}\circ\widetilde{\delta} \tag{A.2.16}$$

One has the long sequence

$$0 \longrightarrow K^0(\mathcal{U},\mathfrak{F}) \xrightarrow{D} K^1(\mathcal{U},\mathfrak{F}) \xrightarrow{D} K^2(\mathcal{U},\mathfrak{F}) \xrightarrow{D} K^3(\mathcal{U},\mathfrak{F}) \xrightarrow{D} \ldots \tag{A.2.17}$$

One denotes by $\mathrm{H}^\bullet(K(\mathcal{U},\mathfrak{F}),D)$ the cohomology of $(K^\bullet(\mathcal{U},\mathfrak{F}),D)$ with respect to the differential operator D.

A.2.1 Isomorphism in cohomology

One establishes the following correspondence. The total complex $K^\bullet(\mathcal{U},\mathfrak{F})$ computes the cohomology of $\Omega^\bullet(\mathsf{A},\mathsf{L})$. more precisely, one has the isomorphism in cohomology:

$$\mathrm{H}^\bullet((\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}}) \simeq \mathrm{H}^\bullet(K(\mathcal{U},\mathfrak{F}),D) \tag{A.2.18}$$

Thus, the cohomology of $(\Omega^\bullet(\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$ is the cohomology of the total complex of the bicomplex $(K^\bullet(\mathcal{U},\mathfrak{F}),D)$. To prove this, one shows that there exist a map $\widetilde{\delta}^* : \mathrm{H}^\bullet((\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}}) \to \mathrm{H}^\bullet(K(\mathcal{U},\mathfrak{F}),D)$ which is bijective. First of all, one proves that such a map exists.

Let ω be a representative element of $H^k((\mathsf{A},\mathsf{L}),D)$. One uses the map $\widetilde{\delta}$ to define $\widetilde{\delta}\omega \in K^k(\mathcal{U},\mathfrak{F})$ consisting in only the top component. As a closed k-form, one write

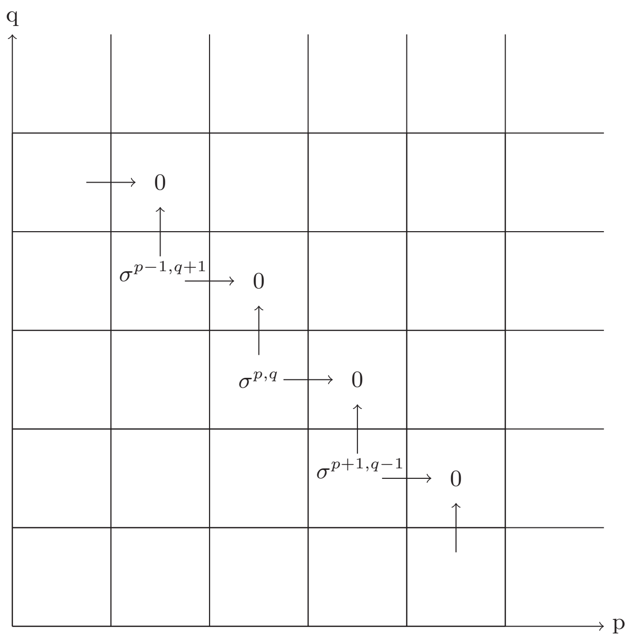
$$D(\widetilde{\delta}\omega) = \widetilde{\delta}\,\widehat{\mathrm{d}}\omega = 0 \tag{A.2.19}$$

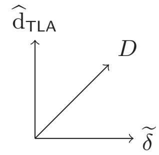
so that the element $\widetilde{\delta}\omega \in K^k(\mathcal{U},\mathfrak{F})$ is also a closed element with respect to D.

Assuming that ω is an exact form on $\Omega(\mathsf{A},\mathsf{L})$, the previous construction would induce that

$$\widetilde{\delta}\omega = \widetilde{\delta}\widehat{\mathrm{d}}\rho = D(\widetilde{\delta}\rho) \tag{A.2.20}$$

so that the element $\widetilde{\delta}\omega \in K^k(\mathcal{U},\mathfrak{F})$ is also an exact element with respect to D. This proves that the map $\widetilde{\delta} : C^{p,q}(\mathcal{U},\mathfrak{F}) \to C^{p+1,q}(\mathcal{U},\mathfrak{F})$ defines a map $\widetilde{\delta}^* : \mathrm{H}^\bullet((\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}}) \to \mathrm{H}^\bullet(K(\mathcal{U},\mathfrak{F}),D)$.





- Let's prove that $\widetilde{\delta}^*$ is surjective. Let σ a representative element of $\mathrm{H}^k(K(\mathcal{U},\mathfrak{F}),D)$. As a closed k-form, it can be written by the sum:

$$\sigma = \sigma^{0,k} + \sigma^{1,k-1} + \ldots + \sigma^{k-1,1} + \sigma^{k,0} \tag{A.2.21}$$

where $\sigma^{i,j} \in C^{i,j}(\mathcal{U},\mathfrak{F})$, such that:

$$\begin{cases} \widehat{\mathrm{d}}_{\mathsf{TLA}}\sigma^{0,k} = 0 \\ \widetilde{\delta}\sigma^{0,k} + \widehat{\mathrm{d}}_{\mathsf{TLA}}\sigma^{1,k-1} = 0 \\ \qquad\vdots \\ \widetilde{\delta}\sigma^{p-1,q+1} + \widehat{\mathrm{d}}_{\mathsf{TLA}}\sigma^{p,q} = 0 \\ \qquad\vdots \\ (-1)^{k-1}\widetilde{\delta}\sigma^{k-1,1} + \widehat{\mathrm{d}}_{\mathsf{TLA}}\sigma^{k,0} = 0 \\ (-1)^k\widetilde{\delta}\sigma^{k,0} = 0 \end{cases} \tag{A.2.22}$$

Since (A.2.11) is exact, the last term $\sigma^{k,0}$ is exact with respect to $\widetilde{\delta}$. Thus, there exist an element $\widetilde{\sigma}^{k-1,0} \in C^{k-1,0}(\mathcal{U}, \mathfrak{F})$ such that

$$(-1)^{k-1}\widetilde{\delta}(\widetilde{\sigma}^{k-1,0}) = \sigma^{k,0} \tag{A.2.23}$$

The second last line of (A.2.22) can now be written as

$$(-1)^{k-1}\widetilde{\delta}\sigma^{k-1,1} + (-1)^{k-1}\widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\delta}(\widetilde{\sigma}^{k-1,0}) = 0$$

One uses the fact that $\widehat{\mathrm{d}}_{\mathsf{TLA}} \circ \widetilde{\delta} = \widetilde{\delta} \circ \widehat{\mathrm{d}}_{\mathsf{TLA}}$ to proves that $\sigma^{k-1,1} + \widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\sigma}^{k-1,0}$ is exact with respect to $\widetilde{\delta}$. Thus, there exist an element $\widetilde{\sigma}^{k-2,1} \in C^{k-2,1}(\mathcal{U}, \mathfrak{F})$ such that

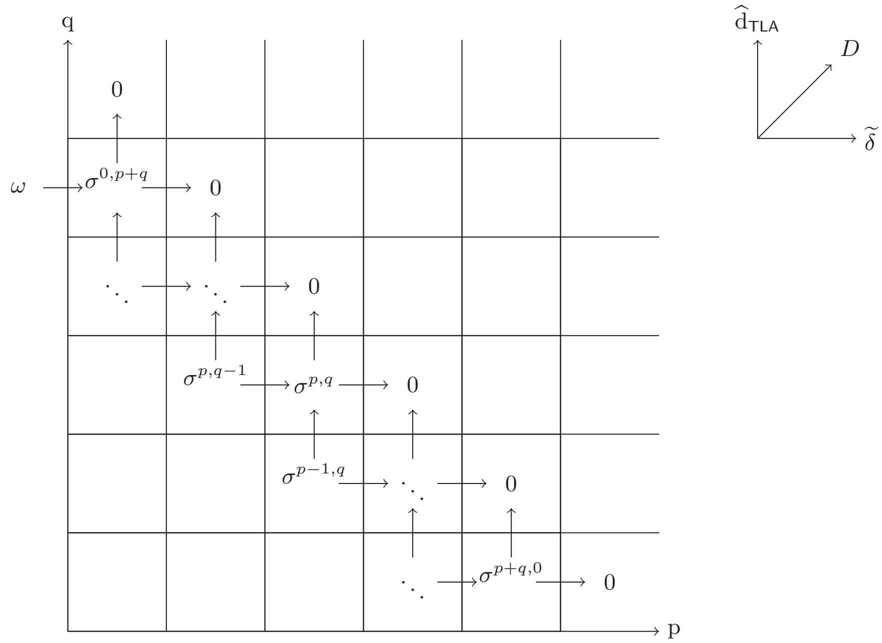
$$(-1)^{k}\widetilde{\delta}(\widetilde{\sigma}^{k-2,1}) - \widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\sigma}^{k-1,0} = \sigma^{k-1,1} \tag{A.2.24}$$

The same procedure is iterate along the diagonal so that one proves that the elements $\sigma^{k-2,2}, \sigma^{k-3,3}, \ldots, \sigma^{0,k}$ can be written in terms of elements $\widetilde{\sigma}^{k-3,2}\widetilde{\sigma}^{k-4,3}, \ldots, \widetilde{\sigma}^{0,k-1}$ and an element $\omega \in \Omega^k(\mathsf{A}, \mathsf{L})$.

More precisely, consider the element $\widetilde{\sigma} \in K^{k-1}(\mathcal{U}, \mathfrak{F})$ composed of the elements $\widetilde{\sigma}^{0,k-1}, \ldots, \widetilde{\sigma}^{k-1,0}$ previously defined. Then, the element $\sigma \in K^k(\mathcal{U}, \mathfrak{F})$ can be written as:

$$\sigma = D(\widetilde{\sigma}) + \widetilde{\delta}\omega \tag{A.2.25}$$

where $\omega \in \Omega^k(\mathsf{A}, \mathsf{L})$.



Obviously, $\sigma - D\widetilde{\sigma}$ is a representative element of $[\sigma] \in \mathrm{H}^k(K(\mathcal{U}, \mathfrak{F}), D)$. It is easy to see that $\widehat{\mathrm{d}}_{\mathsf{TLA}}(\sigma - D(\widetilde{\sigma})) = 0$. Then, $\widehat{\mathrm{d}}_{\mathsf{TLA}} \circ \widetilde{\delta}\omega = \widetilde{\delta} \circ \widehat{\mathrm{d}}_{\mathsf{TLA}}\omega = 0$ so that $\widehat{\mathrm{d}}_{\mathsf{TLA}}\omega = 0$. Thus, the element ω obtained is an element of $Z^k((\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$.

By choosing the representative $\sigma + D(\eta)$ of $[\sigma]$ instead of σ, one would obtain that $\sigma + D(\eta) = D(\widetilde{\sigma} + \eta) + \widetilde{\delta}\omega$. This implies that the induced element ω remains the same. Thus, the element ω does depend only on $[\sigma]$.

Conversely, one assumed that the k-form $\omega \in \Omega^k(\mathsf{A},\mathsf{L})$ defined by the equation (A.2.25) is an exact form. Then, one would obtain

$$\sigma = D(\widetilde{\sigma}) + \widetilde{\delta}\widehat{\mathrm{d}}\rho = D(\widetilde{\sigma} + \widetilde{\delta}\rho) \tag{A.2.26}$$

Thus, the element would be an exact form in $K^k(\mathcal{U},\mathfrak{F})$.

One has shown that, to any $[\sigma] \in H^k(K(\mathcal{U},\mathfrak{F}),D)$, one can associates an element $[\omega] \in H^k((\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}})$. This proves that the map $\widetilde{\delta}^*$ is surjective.

- Let's prove that $\widetilde{\delta}^*$ is injective.

  Let $\omega \in \Omega^k((\mathsf{A},\mathsf{L}))$ such that

$$D(\sigma) = \widetilde{\delta}\omega \tag{A.2.27}$$

  So that $D(\widetilde{\sigma})$ gives 0 except for the component $\widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\sigma}^{0,k-1} \in C^{0,k}(\mathcal{U},\mathfrak{F})$ which is equal to $\widetilde{\delta}\omega$. Implementing the same construction as before, one can easily show that there exist an element $\rho \in \Omega^{k-1}(\mathsf{A},\mathsf{L})$ and an element $\widetilde{\eta}^{0,k-2} \in C^{0,k-2}(\mathcal{U},\mathfrak{F})$ such that

$$\widetilde{\delta}\rho + \widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\eta}^{0,k-2} = \widetilde{\sigma}^{0,k-1} \tag{A.2.28}$$

  Thus, the equation (A.2.27) becomes

$$\widehat{\mathrm{d}}_{\mathsf{TLA}}\widetilde{\delta}\rho = -\widetilde{\delta}\omega \tag{A.2.29}$$

  so that, one obtains that ω is an exact form in $\Omega^k(\mathsf{A},\mathsf{L})$. This proves that the the map $\widetilde{\delta}^*$ maps to the class $[0] \in H^k(K(\mathcal{U},\mathfrak{F}),D)$ only if ω is a representative of the class $[0] \in H^k((\mathsf{A},\mathsf{L}),D)$.

  One has prove that the map $\widetilde{\delta}^*$ is both injective and surjective and thus, one has established an isomorphism of cohomological complexes

$$\mathrm{H}^\bullet((\mathsf{A},\mathsf{L}),\widehat{\mathrm{d}}) \simeq \mathrm{H}^\bullet(K(\mathcal{U},\mathfrak{F}),D) \tag{A.2.30}$$

A.2.2 Associated spectral sequence

Consider the bicomplex $(C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F}),\widehat{\mathrm{d}}_{\mathsf{TLA}},\delta)$. One denotes by $H^{p,q}(C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F}),\widehat{\mathrm{d}}_{\mathsf{TLA}})$ the cohomological class of $C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F})$ with respect to $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ at the bidegrees (p,q). The *spectral sequence associated to the extended bicomplex of Čech-de Rham* $(E_r,\mathrm{d}_r)_{r=1,2}$ is defined as follow. The first term of this spectral sequence is

$$E_1^{p,q} = H^{p,q}(C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F}),\widehat{\mathrm{d}}_{\mathsf{TLA}}).$$

This first term can also be written as:

$$E_1^{p,q} = C^p(\mathfrak{U},\mathcal{H}^q) \tag{A.2.31}$$

where $\mathcal{H}^q$ is the presheaf which associates to an open subset $\mathcal{U}_{\alpha_0\ldots\alpha_p} \subset \mathcal{M}$ the cohomology space $\mathcal{H}^q(\mathcal{U}_{\alpha_0\ldots\alpha_p}) = \mathrm{H}^q(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_{\alpha_0\ldots\alpha_p},\mathfrak{g}),\widehat{\mathrm{d}}_{\mathsf{TLA}})$. The differential operator d_1 is induced from $\widetilde{\delta}$ as follow:

$$\mathrm{d}_1 : H^{p,q}(C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F}),\widehat{\mathrm{d}}_{\mathsf{TLA}}) \to H^{p+1,q}(C^{\bullet,\bullet}(\mathcal{U},\mathfrak{F}),\widehat{\mathrm{d}}_{\mathsf{TLA}}) \quad ; \quad \mathrm{d}_1\omega = \prod_{\alpha_0<\ldots<\alpha_p}(\widetilde{\delta}\omega)_{\alpha_0\ldots\alpha_p} \tag{A.2.32}$$

where ω is a representative of $[\omega] \in \mathrm{H}^{p,q}(C^{\bullet,\bullet}(\mathcal{U}, \mathfrak{F}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$. It is straightforward to see that the image of ω by d_1 is representative of the an element of $\mathrm{H}^{p+1,q}(C^{\bullet,\bullet}(\mathcal{U}, \mathfrak{F}), \widehat{\mathrm{d}}_{\mathsf{TLA}})$.

The second term of this spectral sequence is then

$$E_2^{p,q} = \mathrm{H}^p(E_1^{p,q}, \mathrm{d}_1) = \mathrm{H}^p(C^\bullet(\mathcal{U}, \mathcal{H}^q), \widetilde{\delta})$$

This is a Leray-Serre spectral sequence for which the fibration is not just along ordinary spaces but along differential structures.

Recall that the differential operator $\widehat{\mathrm{d}}_{\mathsf{TLA}}$ can be locally decomposed as $\widehat{\mathrm{d}}_{\mathsf{TLA}} = \mathrm{d} + \mathrm{s}'$ where d is the de Rham derivative and s' is the Chevalley-Eilenberg derivative equipped with the adjoint representation of $\mathfrak{g}$. Morerover, because $\mathcal{U}$ is a good cover, the de Rham cohomology vanishes for $q \neq \dim(\mathcal{M})$. Then, one has

$$\mathrm{H}^q(\Omega^\bullet_{\mathsf{TLA}}(\mathcal{U}_{\alpha_0 \dots \alpha_p}, \mathfrak{g}), \widehat{\mathrm{d}}_{\mathsf{TLA}}) = \mathrm{H}^q((\mathfrak{g}, \mathfrak{g}), \mathrm{s}'),$$

the Lie algebra cohomology of the usual differential complex $(\wedge^\bullet \mathfrak{g}^* \otimes \mathfrak{g}, \mathrm{s}')$.

One has proved that there exist a spectral sequence $(E_r, \widehat{\mathrm{d}}_r)_{r \geq 0}$ which abouts to the cohomology of $(\Omega^\bullet(\mathsf{A}, \mathsf{L}), \widehat{\mathrm{d}})$ such that $E_2^{p,q} = \mathrm{H}^p(\mathcal{M}; \mathcal{H}^q(\mathfrak{g}, \mathfrak{g}))$.

Remember that the restriction map $i_{\mathcal{U}_i}^{\mathcal{U}_j}$ for the presheaf $\mathfrak{F}$ makes use of the action of $\widehat{\alpha}_{\mathcal{U}_i}^{\mathcal{U}_j}$. The induced restriction map for the presheaf $\mathcal{H}^q$ is obtained by the induced action of $\widehat{\alpha}_{\mathcal{U}_i}^{\mathcal{U}_j}$ in cohomology. This implies that the presheaf $\mathcal{H}^q$ is not necessarily a constant presheaf.

Appendix B

Code Mathematica

```
(*CONSTRUCTION su(N) PAR RECURRENCE *)
(*donnée de la dimension*)

rankN = 3;
dimN = rankN^2 - 1;
(*initialisation*)
Esu2 = Table[0, {i, 1, 3}, {j, 1, 2}, {k, 1, 2}];
MatrixForm[Esu2[[1]] = I/Sqrt[2] {{0, 1}, {1, 0}}];
MatrixForm[Esu2[[2]] = I/Sqrt[2] {{0, -I}, {I, 0}}];
MatrixForm[Esu2[[3]] = I/Sqrt[2] {{1, 0}, {0, -1}}];
Esuprev = Esu2;
(*relation de récurrence*)
For[
  r = 2, r < rankN, r++,
  dim = r^2 - 1;
  Esunext =
   Table[0, {i, 1, (r + 1)^2 - 1}, {j, 1, (r + 1)}, {k, 1, (r + 1)}];
  (*inclusion d'un bloc su(N-1) dans su(N)*)
  Do[
    Do[
     If[i < (r + 1) && j < (r + 1) ,
      Esunext[[k, i, j]] = Esuprev[[k, i, j]], Esunext[[k, i, j]] = 0
      ]
     , {i, 1, (r + 1)}, {j, 1, (r + 1)}
     ]
    , {k, 1, dim}
    ]
   (*rajout des termes sur les bords*)
   Do[
    Do[
     If[EvenQ[k],
      Esunext[[(dim + 2 m + k), (r + 1), (1 + m)]] = -Sqrt[(1/2)],
      Esunext[[(dim + 2 m + k), (r + 1), (1 + m)]] = I Sqrt[1/2]]
     , {k, 1, 2}]
    , {m, 0, (r - 1)}];
  Do[
   Do[
    If[EvenQ[k],
     Esunext[[(dim + 2 m + k), (1 + m), (r + 1)]] = Sqrt[1/2],
```

```
     Esunext[[(dim + 2 m + k), (1 + m), (r + 1)]] = I Sqrt[1/2]]
    , {k, 1, 2}]
   , {m, 0, (r - 1)}];
  (*dernier générateur*)

  Esunext[[((r + 1)^2 - 1), (r + 1), (r + 1)]] = (-I r)/
   Sqrt[(r + 1) r];
  Do[
    Esunext[[((r + 1)^2 - 1), i, i]] = I/Sqrt[(r + 1) r]
    , {i, 1, r}
    ]
   (*fin de la boucle*)
   Esuprev = Esunext;
  ];
EsuN = Esunext;
(*impression des générateurs de l'algèbre de Lie suN*)
Do[
  Print[
   MatrixForm[EsuN[[i]]]
   ], {i, 1, dimN}
  ];
(* Vérification Tr = -\delta *)

TrN = Table[0, {i, 1, dimN}, {j, 1, dimN}];
Do[
  TrN[[a, b]] = Tr[
    EsuN[[a]].EsuN[[b]]
    ], {a, 1, dimN}, {b, 1, dimN}
  ];

Print[MatrixForm[TrN]];




(******(*calcul des constantes de structures pour su(N)*)************)

CoefN = Table[0, {i, 1, dimN}, {j, 1, dimN}, {k, 1, dimN}];
Do[
  CoefN[[a, b, c]] = Tr[
    (EsuN[[a]].EsuN[[b]] - EsuN[[b]].EsuN[[a]]).(EsuN[[c]])
    ], {a, 1, dimN}, {b, 1, dimN}, {c, 1, dimN}
  ];

(****************************************)
(*impression des constantes de structures*)
(****************************************)
```

```
(* simplification de l'écriture des constantes de structures*)
Do[
  CoefN[[i]] = Simplify[CoefN[[i]]]
  , {i, 1, dimN}
  ];
(*impresion des constantes de structures*)
Do[
  Print[MatrixForm[CoefN[[i]]]]
  , {i, 1, dimN}
  ];



(*calcul de la matrice des masses avec \tau=1*)

(*définition du projecteur \tau*)

tau = Table[0, {i, 1, dimN}, {j, 1, dimN}];
Do[
  tau[[i, i]] = f[i]
  , {i, 1, dimN}
  ];
(*réinitialisation \tau*)
Do[
  f[i] = 1
  , {i, 1, dimN}
  ];
(* fin réinitialisation  \tau*)
MatrixForm[tau]
(*calcul du coefficient de masse*)
MasssuN = Table[
   Sum[
    CoefN[[a0, b0, c]] CoefN[[a1, b1, c]] tau[[d, b0]] tau[[d, b1]]
    , {b0, 1, dimN}, {b1, 1, dimN}, {c, 1, dimN}, {d, 1, dimN}
    ]
   , {a0, 1, dimN}, {a1, 1, dimN}
   ];
(*affichage*)
MatrixForm[MasssuN]
```

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Résumé de la thèse

Connus des mécaniciens de la géométrie de Poisson, les algébroïdes de Lie transitifs sont ici étudiés du point de vue de leurs sections afin de développer un formalisme algébrique plus proche de celui développé par les théories de jauge. Dans cette approche, les algébroïdes de Lie transitifs s'apparentent à une généralisation des champs de vecteurs sur la variété de base. Ce mémoire de thèse a pour objet l'étude des connexions généralisées sur les algébroïdes de Lie transitifs et la construction de théories de jauge.

Les connexions ordinaires sur les algébroïdes de Lie transitifs sont définies par des 1-formes de connexion de l'algébroïde de Lie à valeurs dans son noyau et vérifiant une contrainte de normalisation sur ce noyau. En relâchant cette contrainte, on construit l'espace des 1-formes de connexions généralisées qui se décomposent, à l'aide d'une connexion ordinaire de fond, comme la somme d'une connexion ordinaire et d'un paramètre purement algébrique défini sur le noyau.

Dans l'esprit des théories Yang-Mills, une action invariante de jauge est définie comme la “norme” de la courbure associée à une connexion généralisée. De cette action, il découle un lagrangien composé des termes des théories de jauge de type Yang-Mills-Higgs : le terme cinétique associé aux champs de jauge et le terme de couplage minimal pour un champ tensoriel scalaire plongé dans un potentiel quartique.

Dans le cas des algébroïdes de Lie d'Atiyah, la réduction du groupe de symétrie de la théorie s'effectue par une redistribution des degrés de liberté dans l'espace fonctionnel des champs de la théorie. Il résulte de ces manipulations la définition d'une théorie de type Yang-Mills dont les bosons vecteurs sont des champs massifs.

Abstract Thesis

Transitive Lie algebroids are usually studied from the point of view of the geometry of Poisson. Here, they are preferentially defined in terms of sections of fiber bundle in order to get close to the formalism of the gauge field theory. Then, transitive Lie algebroids can be seen as a generalization of vector fields on the base manifold. This PhD thesis is concerned with the study of generalized connections on transitive Lie algebroids and the construction of gauge theories.

Ordinary connections on transitive Lie algebroids are defined as the subset of 1-forms on Lie algebroids with values in its kernel which fulfill a normalization constraint on this kernel. By relaxing this constraint, we build the space of generalized connection 1-forms. Using a background connection, we show that any generalized connections can be decomposed as the sum of an ordinary connection and a purely algebraic parameter defined on the kernel.

As in Yang-Mills theories, we define a gauge invariant functional action as the “norm” of the curvature associated to a generalized connection. Then, the Lagrangian associated to this action forms a Yang-Mills-Higgs type model composed with the field strength associated to gauge fields and a minimal coupling with a tensorial scalar field embedded into a quartic potential.

In the case of Atiyah Lie algebroids, the symmetry group of the theory can be reduced by using an appropriate rearrangement of the degrees of freedom in the functional space of fields. We thus obtain a Yang-Mills type theory describing massive vector bosons.